Chapter 3. Modeling in the Time Domain

Things to know

- State-space representation

- Conversion from a transfer function and a state-space representation and vice versa

- Getting solutions using a state equation

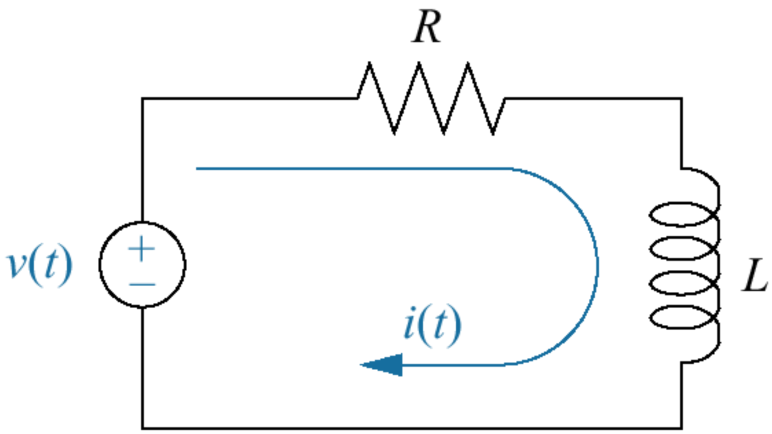


Figure 3.1 *RL* network

By KVL

$$L\frac{di}{dt}+Ri=v(t) \qquad (3.1)$$

Taking LT yields first order state equation

$$L\left[sI(s)-i(0)\right]+RI(s)=V(s) \qquad (3.2)$$

$$(Ls+R)I(s)=V(s)+Li(0)$$

$$If\ v(t)=u(t): unit\ step\ fn., \quad V(s)=\frac{1}{s}$$

$$I(s)=\left(\frac{1/L}{s+R/L}\cdot\frac{1}{s}\right)+\frac{i(0)}{s+R/L}$$

$$=\frac{1}{R}\left(\frac{1}{s}-\frac{1}{s+R/L}\right)+\frac{i(0)}{s+R/L}$$

$$\therefore i(t)=\frac{1}{R}\left(1-e^{-(R/L)t}\right)+i(0)e^{-(R/L)t}$$

From (3.1)

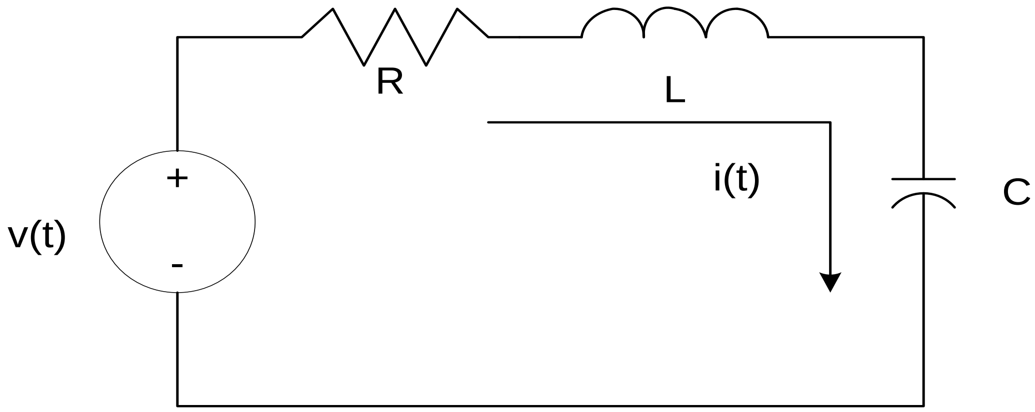
$$\frac{di(t)}{dt} = \dot{i}(t) = -\frac{R}{L} i(t) + \frac{v(t)}{L} \quad \Rightarrow \quad \textit{State equation}$$

$$v_R(t) = Ri(t) \quad \Rightarrow \quad \textit{Output equation}$$

or

$$v_L(t) = -Ri(t) + v(t) \quad \Rightarrow \quad \textit{Output equation}$$

Ex.2)



$$L\frac{di}{dt}+Ri+\frac{1}{C}\int idt=v(t) \qquad (1)$$

$$\frac{dq}{dt}=i(t) \qquad (2)$$

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=v(t) \qquad (3)$$

Since the system is a second order

state equations

$$
\left[\begin{aligned}
\frac{dq}{dt} &= i \\
\frac{di}{dt} &= -\frac{1}{LC}q - \frac{R}{L}i + \frac{1}{L}v(t)
\end{aligned}\right.
$$

in matrix form

$$
\begin{bmatrix} \dfrac{dq}{dt} \\ \dfrac{di}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{LC} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} q \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} v(t)
$$

$$
\dot{x} = A x + B u
$$

$from$ (1)

$$
v_L(t) = -\frac{1}{C}q(t) - Ri(t) + v(t)
$$

in matrix form for output equation

$$
\left[v_L(t)\right] = \begin{bmatrix} -\dfrac{1}{C} & -R \end{bmatrix} \begin{bmatrix} q \\ i \end{bmatrix} + [1]\, v(t)
$$

$$
y = C x + D u
$$

State variable form

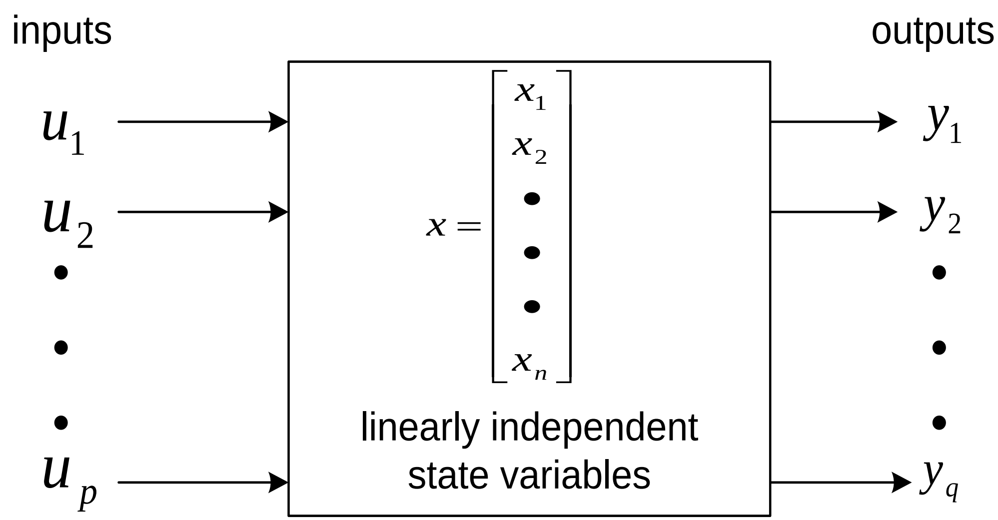
State variables (상태변수)
: (Linearly independent) variables that can completely define the behavior of the system.

Output variable (출력변수): A variable(s) that can be measured.

Note: In the RLC circuit example, q and i should be linearly independent.
Q: What happens if v and i are chosen?
Answer: Will not work because v and i are linearly dependent, that is, v = R i.

Q: A state representation is unique?

General state-space representation



Dynamic equations

- state equation $\quad \dot{x} = Ax + Bu$
- output equation $\quad y = Cx + Du$

where,

x=state vector $(n \times 1)$

$$\dot{x} = derivative \;\; of \;\; x = \frac{dx(t)}{dt} \;(n \times 1)$$

y= output vector $(q \times 1)$

u= input or control vector $(p \times 1)$

A= system matrix $(n \times n)$

B= input coupling matrix $(n \times p)$

C= output matrix $(q \times n)$

D= feedforward matrix $(q \times p)$

Converting a D.E. to a state equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u(t)$$

$$x_1 = y$$

$$x_2 = \frac{dy}{dt} = \dot{x}_1$$

$$x_3 = \frac{d^2 y}{dt^2} = \dot{x}_2 = \ddot{x}_1$$

•

•

•

$$x_n = \frac{d^{n-1} y}{dt^{n-1}} = \dot{x}_{n-1}$$

$$\frac{d^n y}{dt^n} = \dot{x}_n = -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n + b_0 u(t)$$

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \bullet \\ \bullet \\ \bullet \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \bullet & \bullet & \bullet & 0 \\ 0 & 0 & 1 & 0 & \bullet & \bullet & \bullet & 0 \\ 0 & 0 & 0 & 1 & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \bullet \\ \bullet & \bullet & \bullet & \bullet & & \bullet & & \bullet \\ \bullet & \bullet & \bullet & \bullet & & & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet & \bullet & \bullet & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \bullet & \bullet & \bullet & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \bullet \\ \bullet \\ \bullet \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bullet \\ \bullet \\ \bullet \\ 0 \\ b_0 \end{bmatrix} u(t)
$$

output equations

$$
y = \begin{bmatrix} 1 & 0 & 0 & \bullet & \bullet & \bullet & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \bullet \\ \bullet \\ \bullet \\ x_{n-1} \\ x_n \end{bmatrix}
$$

Converting a transfer function to a state equation

$$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

Assuming zero initial conditions, convert the transfer function to a differential equation as follows:

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r(t)$$

$$x_1 = c$$

$$x_2 = \dot{c} = \dot{x}_1$$

$$x_3 = \ddot{c} = \dot{x}_2$$

$$\dot{x}_3 = \dddot{c} = -9\ddot{c} - 26\dot{c} - 24c + 24r$$

$$= -24x_1 - 26x_2 - 9x_3 + 24r$$

$$y = c = x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r(t)$$

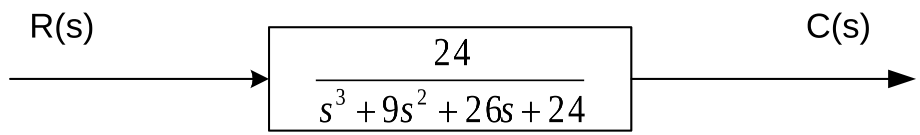
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

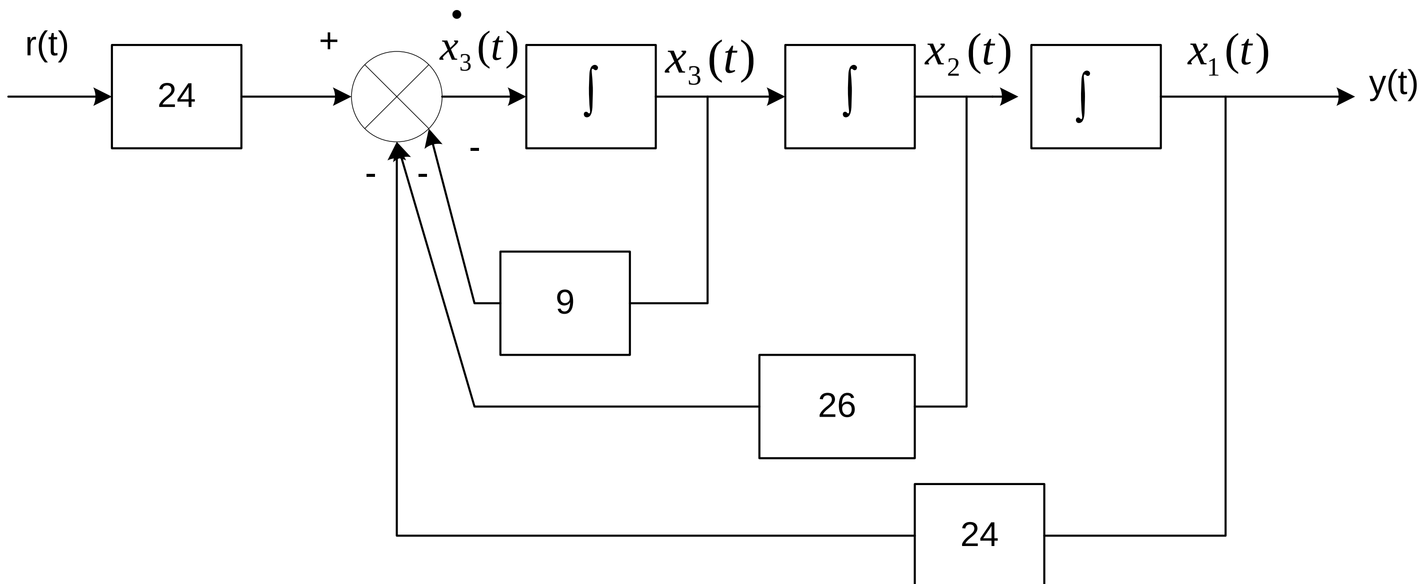
Block diagram

$$\frac{C(s)}{R(s)}=\frac{24}{s^3+9s^2+26s+24}$$

$$\dddot{c}=-9\ddot{c}-26\dot{c}-24c+24r$$

$$=-24x_1-26x_2-9x_3+24r$$





$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 24\end{bmatrix}r(t)$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Converting a transfer function with a polynomial numerator

Assume $n \geq m$

$$R(s) \longrightarrow \boxed{\dfrac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}} \longrightarrow C(s)$$

$$R(s) \longrightarrow \boxed{\dfrac{1}{s^n + \frac{a_{n-1}}{a_n} s^{n-1} \cdots + \frac{a_1}{a_n} s + \frac{a_0}{a_n}}} \xrightarrow{x_1(s)} \boxed{\frac{b_m}{a_n} s^m + \frac{b_{m-1}}{a_n} s^{m-1} + \cdots \frac{b_1}{a_n} s + \frac{b_0}{a_n}} \longrightarrow C(s)$$

Internal variables : $x_2(s), x_3(s) \cdots$

$$Y(s) = C(s) = \left(\frac{b_m}{a_n} s^m + \frac{b_{m-1}}{a_n} s^{m-1} + \cdots \frac{b_1}{a_n} s + \frac{b_0}{a_n}\right) x_1(s)$$

Taking ILT with zero initial conditions

$$y(t) = \frac{b_m}{a_n} \frac{d^m x_1}{dt^m} + \frac{b_{m-1}}{a_n} \frac{d^{m-1} x_1}{dt^{m-1}} + \cdots + \frac{b_1}{a_n} \frac{dx_1}{dt} + \frac{b_0}{a_n} x_1$$

$$y(t) = \frac{b_0}{a_n} x_1 + \frac{b_1}{a_n} x_2 + \cdots + \frac{b_{m-1}}{a_n} x_m + \frac{b_m}{a_n} x_{m+1}$$

The output equation is

$$y = \begin{bmatrix} \frac{b_0}{a_n} & \frac{b_1}{a_n} & \cdot & \cdot & \cdot & \frac{b_m}{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{m+1} \end{bmatrix}$$

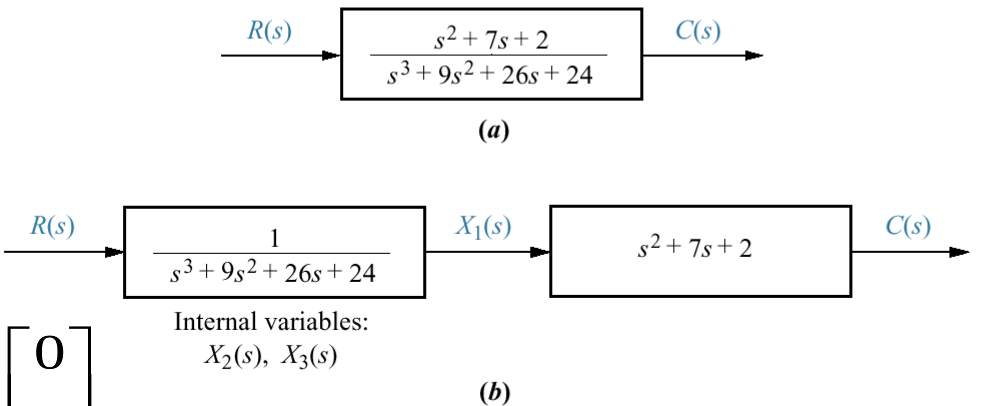
The state equation is the same as before

$$r(t) = x_1^{(n)}(t) + \frac{a_{n-1}}{a_n} x_1^{(n-1)}(t) + \cdots + \frac{a_1}{a_n}\dot{x}_1(t) + \frac{a_0}{a_n} x_1(t)$$

$$\begin{aligned} x_1 &= x_1, \\ \dot{x}_1 &= \dot{x}_1 = x_2 \\ \dot{x}_2 &= \ddot{x}_1 = x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_1^{(n-1)} = x_n \\ \dot{x}_n &= x_1^{(n)} = -\frac{a_0}{a_n}x_1 - \frac{a_1}{a_n}x_2 - \cdots - \frac{a_{n-1}}{a_n}x_n + r \end{aligned}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \cdot & \cdot & \cdot & -\frac{a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} r$$

Example: Converting a transfer function with a polynomial numerator



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \qquad (3.63)$$

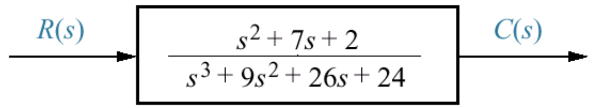
$$C(s) = \left(b_2 s^2 + b_1 s + b_0\right) X_1(s) = (s^2 + 7s + 2) X_1(s) \qquad (3.64)$$

$$c = \ddot{x}_1 + 7\dot{x}_1 + 2x_1 \qquad (3.65)$$

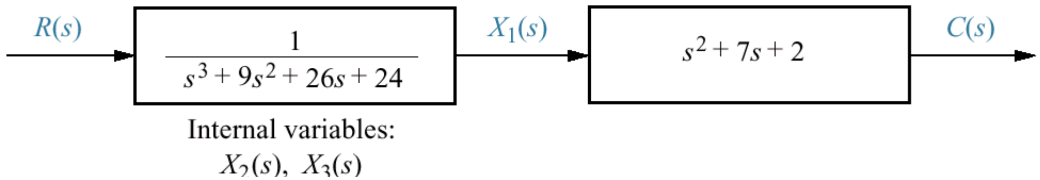
$$x = x_1 \ , \ \dot{x}_1 = x_2 \ , \ \ddot{x}_1 = x_3$$

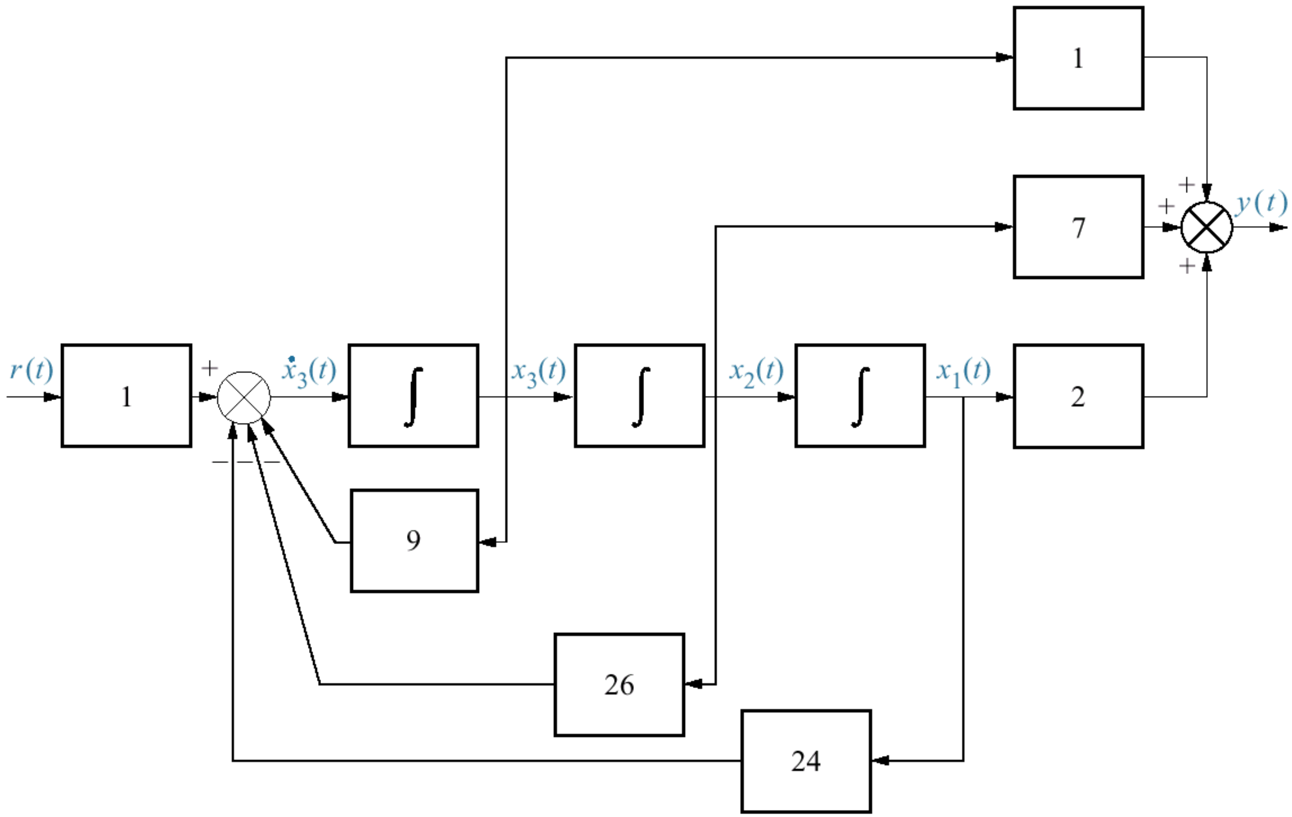
$$y = c(t) = b_2 x_3 + b_1 x_2 + b_0 x_1 = x_3 + 7x_2 + 2x_1 \qquad (3.66)$$

$$y = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (3.67)$$


R(s)
s^2 + 7s + 2 / s^3 + 9s^2 + 26s + 24
C(s)

(a)


R(s)
1 / s^3 + 9s^2 + 26s + 24
X1(s)
s^2 + 7s + 2
C(s)
Internal variables:
X2(s), X3(s)


r(t)
1
+
ẋ3(t)
∫
x3(t)
∫
x2(t)
∫
x1(t)
2
7
1
y(t)
9
26
24

(c)

Converting a state equation to a transfer function

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Taking the Laplace transform with zero initial conditions yields

$$sX(s) = AX(s) + BU(s), \quad Y(s) = CX(s) + DU(s)$$

$$(sI - A)X(s) = BU(s)$$

$$or\ X(s) = (sI - A)^{-1}BU(s)$$

where I is the identity matrix. Hence

$$\therefore \quad Y(s) = C(sI - A)^{-1}BU(s) + DU(s)$$

$$= [C(sI - A)^{-1}B + D]U(s)$$

$$Y(s) = T(s)U(s)$$

where

$$T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D \quad : \text{transfer function}$$

Example: Converting a state equation to a transfer function

Find $T(s) = \dfrac{Y(s)}{R(s)}$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u(t), \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

$$sI - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{adj(sI - A)}{\det(sI - A)} = \frac{\begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3+3s^2+2s+1}$$

$$\det(sI - A) = s\begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ 1 & s+3 \end{vmatrix} + 0\begin{vmatrix} 0 & s \\ 1 & 2 \end{vmatrix} = s^3+3s^2+2s+1$$

$$T(s) = C(sI - A)^{-1}B = \frac{10(s^2+3s+2)}{s^3+3s^2+2s+1}$$

$$\textit{cofactor} \quad c_{ij} = (-1)^{i+j} M_{ij} \quad \text{(signed minor)}$$

$$Adj\ A = \begin{bmatrix} c_{11} & c_{12} & \cdot & \cdot & \cdot & c_{1n} \\ c_{21} & c_{22} & \cdot & \cdot & \cdot & c_{2n} \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ c_{n1} & c_{n2} & \cdot & \cdot & \cdot & c_{nn} \end{bmatrix}^T \quad (Adj\ A = c_{ij}{}^T)$$

$$T(s) = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1}$$

Solving the state equation using the Laplace transform

Taking the Laplace form

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$X(s) = [sI - A]^{-1}[x(0) + BU(s)] \qquad (1)$$

$$Y(s) = CX(s) + DU(s) \qquad (2)$$

$(1) \longrightarrow (2)$

$$Y(s) = C[sI - A]^{-1}[x(0) + BU(s)] + DU(s)$$

When $x(0) = 0$,

$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

The transfer function is defined as

$$T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

or $T(s) = C\left[\dfrac{adj(sI - A)}{\det(sI - A)}\right]B + D$

or $T(s) = \dfrac{C\ adj(sI - A)B + D\det(sI - A)}{\det(sI - A)}$ (3)

Example

$$Ex)\ A=\begin{bmatrix}-5 & -6\\ 1 & 0\end{bmatrix},\quad B=\begin{bmatrix}1\\ 0\end{bmatrix},\quad C=\begin{bmatrix}0 & 1\end{bmatrix},\quad D=0$$

$$\dot{x}=Ax+Bu$$

$$y=Cx+Du$$

$$Find\ T(s)\ and\ t(t):\quad T(s)=C(sI-A)^{-1}B+D$$

$$T(s)=\frac{\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}s & 1\\ -6 & s+5\end{bmatrix}^{T}\begin{bmatrix}1\\ 0\end{bmatrix}}{s^2+5s+6}=\frac{1}{(s+2)(s+3)}$$

$$t(t)=L^{-1}\{T(s)\}=L^{-1}\left\{\frac{1}{s+2}-\frac{1}{s+3}\right\}$$

$$t(t)=e^{-2t}-e^{-3t}$$

Poles, zeros, and eigen values

Transfer function $T(s) = \frac{N(s)}{D(s)} = C(sI - A)^{-1}B + D$

Poles: the roots of $D(s) = 0$

Zeros: the roots of $N(s) = 0$

Eigen values: the roots of $\det(sI\text{-}A) = 0$ $\Rightarrow$ poles

Definition: A zero vector x_i that satisfies $Ax_i = \lambda_i x_i$ is called the eigen vector of the matrix A associated with the eigen value λ_i

Note: λ_i is called a natural frequency (or an eigenvalue) of the system $\dot{x} = Ax$.

Why: Assume that $x(t) = e^{\lambda_i t} x_0$, where x_0 is an initial condition. Then

$$\dot{x}(t) = \lambda_i e^{\lambda_i t} x_0 = Ax = Ae^{\lambda_i t} x_0$$

$$\lambda_i x_0 = Ax_0$$

$$(\lambda_i I - A)x_0 = 0 \qquad (1)$$

Hence, λ_i is an eigenvalue and x_0 is the corresponding eigenvector of the matrix A.

For a nonzero x_0, (1) is satisfied if and only if $\det(\lambda_i I - A) = 0$.

A zero is a value of frequency s such that the input is $u(t) = u_0 e^{st}$ then the output is zero. That is, the system has a nonzero input signal, but nothing comes out.

$$Let\ \ x(t) = e^{st} x_0$$

$$\dot{x} = Ax + Bu$$

$$\dot{x} = s x_0 e^{st} = A x_0 e^{st} + B u_0 e^{st}$$

$$[(sI - A)x_0 - Bu_0]e^{st} = 0$$

$For\ a\ nontrivial\ solution$(비자명해),

$$e^{st} \neq 0$$

$$then, (sI - A)x_0 - Bu_0 = 0 \qquad (2)$$

$$y = Cx + Du$$

$$= Ce^{st} x_0 + Du_0 e^{st} = 0$$

$$(Cx_0 + Du_0)e^{st} = 0$$

$$Since\ \ e^{st} \neq 0, \quad Cx_0 + Du_0 = 0 \qquad (3)$$

$combining$ (2) and (3) $yields$

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$For\ a\ nontrivial\ solution, \quad \begin{vmatrix} sI - A & -B \\ C & D \end{vmatrix} = 0$$

$$\det\begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} = 0$$

The roots are zeros.

$$\therefore T(s) = \frac{\det\begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix}}{\det[sI-A]}$$

Ex.) Given that

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$A = \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

The zeros are such that

$$\det\begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} = \begin{vmatrix} s+3 & -1 & -1 \\ 0 & s & -2 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

$$= (s+3)\begin{vmatrix} s & -2 \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & s \\ 1 & 0 \end{vmatrix}$$

$$= 0 + 2 + s = s + 2 = 0$$

$$zero\ is\ s = -2$$

$$Poles\ are\ s = 0,\ \ \text{-}3 \ \Leftarrow \det[sI - A] = \begin{vmatrix} s+3 & -1 \\ 0 & s \end{vmatrix} = s(s+3)$$

$$T(s) = \frac{s+2}{s(s+3)}$$

Previously

$$\mathrm{T(s)} = \mathrm{C}\left[\frac{\mathrm{adj(sI\text{-}A)}}{\det(sI - A)}\right]B + D$$

$$= \frac{1}{s(s+3)}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & 1 \\ 0 & s+3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} s & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}}{s(s+3)} = \frac{s+2}{s(s+3)} \ \Rightarrow The\ same\ result.$$

Frequency Domain vs. Time Domain

1. Frequency domain (classical approach)

- Only for linear time-invariant (LTI) SISO systems
- Easy understanding (graphical analysis possible)
- Both zeros and poles obtainable
- Cannot predict the behavior of higher orders poles (usually for up to second-order poles)

2. Time domain or state-space domain (Modern approach)

- Unified method
- Can be extended to nonlinear, time-varying systems
- Multi-input, multi-output systems (MIMO)
- Easy computer installation
- May be sensitive to parameter changes (no specification of closed-loop zeros)

● <u>제어시스템을 설계하는 목적</u>

- 동적 시스템의 성능을 보다 향상 ==> 피드백 제어시스템을 사용
- 성능-강인성(performance- robustness)을 유지
- 항상 안정도 문제를 고려

● <u>피드백 제어시스템의 성능과 안정도</u>

▶제어시스템의 성능: 주파수 및/또는 시간 역에서 해석

- 시스템의 주파수응답:
  입력주파수에 따른 피드백 제어시스템의 성능을 파악

- 시간역에서의 시스템 성능:
  시스템의 과도응답과 정상상태응답 => 제어시스템의 성능을 예측

▶ 제어시스템의 안정도: 폐루프 시스템의 절대적 안정

- 공칭안정도(nominal stability), 절대안정도(absolute stability)

시스템의 수학적 모델에 대해 안정한지 불안정한지를 논함

✓ 시스템의 특성방정식의 근을 직접 조사하는 방법

$\Rightarrow$ Routh 안정도 판별법(Routh's stability criterion)

근궤적법, Nyquist 안정도 판별법(Nyquist stability criterion)

- 상대안정도(relative stability) : 모델의 불확실성에 대해서 상대적으로

어느 정도 더 안정한지를 논함.

모델의 불확실성에 대해서도 안정도-강인성을 유지

- MATLAB을 이용, 제어시스템의 성능과 안정도를 평가하고 SIMULINK를

활용하여 제어시스템의 시간응답을 시뮬레이션

◆ 제어시스템의 기본 정의

시스템 입력: 기준입력 *r(s)*, 외란 *d(s)*, 센서잡음 *n(s)*

플랜트의 출력: *y(s)*

G(s): 플랜트 전달함수, K(s): 제어기 전달함수

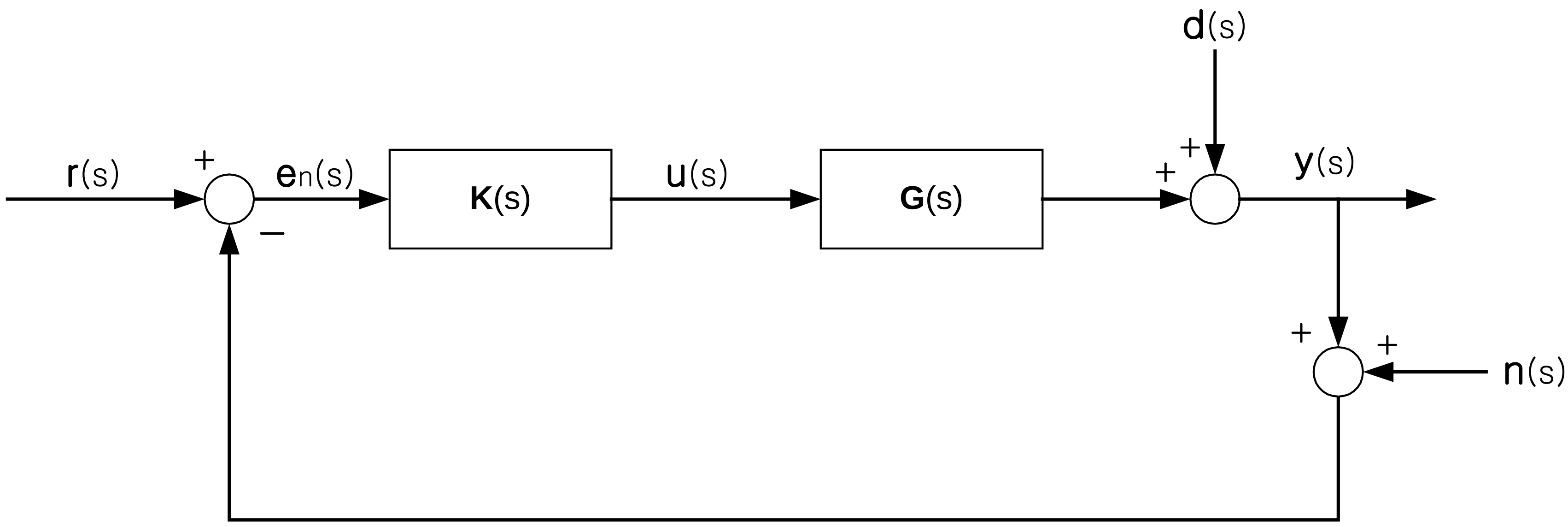
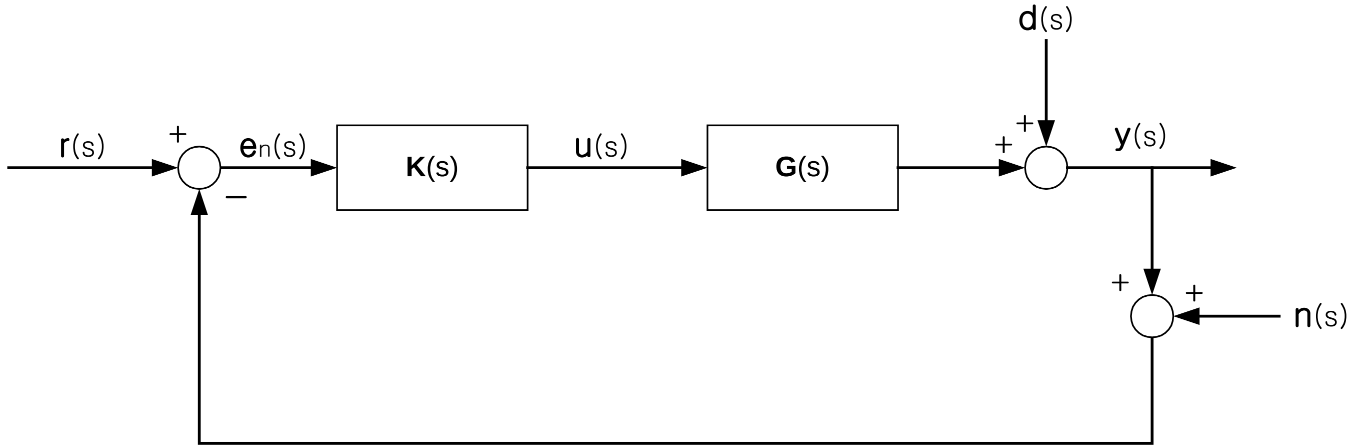


그림 3.1 일반적인 피드백 제어시스템



출력 y(s)는

$$y(s) = y_r(s) + y_d(s) + y_n(s)$$

$$y_r(s) = \frac{G(s)K(s)}{1+G(s)K(s)} r(s)$$

$$y_d(s) = \frac{1}{1+G(s)K(s)} d(s)$$

$$y_n(s) = -\frac{G(s)K(s)}{1+G(s)K(s)} n(s)$$

출력 y(s)는

$$y(s)=\frac{G(s)K(s)}{1+G(s)K(s)}r(s)+\frac{1}{1+G(s)K(s)}d(s)-\frac{G(s)K(s)}{1+G(s)K(s)}n(s) \qquad (3.1)$$

−루프(loop) 전달함수: G(s)K(s)

− 귀환차(return difference) 전달함수: 1+G(s)K(s)

− 감도(sensitivity) 전달함수: $S(s)=\frac{1}{1+G(S)K(s)}$ (3.2)

− 여감도(complementary sensitivity) 혹은 폐루프 전달함수:

$$T(s)=\frac{G(s)K(s)}{1+G(s)K(s)} \qquad (3.3)$$

- 실제 추적오차(tracking error) e(s):

$$e(s) = r(s) - y(s) \tag{3.4}$$

$$= \underbrace{\frac{1}{1+G(s)K(s)}}_{=S(s)}\{r(s) - d(s)\} + \underbrace{\frac{G(s)K(s)}{1+G(s)K(s)}}_{=T(s)}n(s) \tag{3.5}$$

혹은, $e(s) = S(s)\left\{r(s) - d(s)\right\} + T(s)n(s)$ (3.6)

다음 구속조건이 만족되어야 하므로 동시에 S(s) = 0, T(s) = 0은 불가능하다.

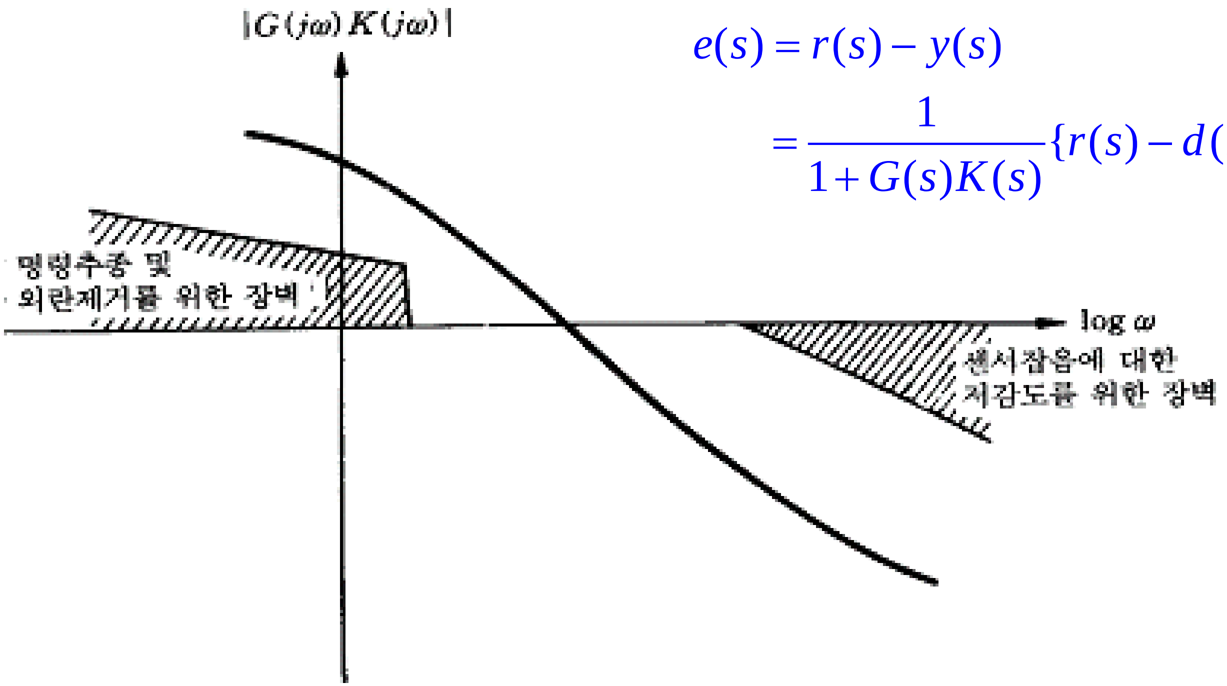
$$\because\ S(s) + T(s) = 1 \tag{3.7}$$

- 기준입력 r(s)와 외란 d(s)는 저주파에서 에너지를 가짐
  ==> 저주파에서는 감도 전달함수 S(s)를 작게함
- 센서잡음 n(s)는 고주파에서 에너지를 가짐
  ==> 고주파에서는 여감도 전달함수 T(s)를 작게함

▶ 센서잡음에 대한 저감도

추적오차식 (3.5)에서 센서잡음 n(s)에 대한 영향만을 고려하면

$$e(s)=\frac{G(s)K(s)}{1+G(s)K(s)}n(s) \qquad (3.34)$$



$$\begin{aligned} e(s) &= r(s)-y(s) \\ &= \frac{1}{1+G(s)K(s)}\{r(s)-d(s)\}+\frac{G(s)K(s)}{1+G(s)K(s)}n(s) \end{aligned}$$

그림 3.5 일반적인 제어시스템의 루프 형상

[예제 2.21] MATLAB을 이용하여 다음과 같은 상태공간 모델식을 전달함수로 표현하고, 시스템의 극점을 구하기로 한다.

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} -5 & 1 & 0 \\ -3 & 0 & 1 \\ -4 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

구한 전달함수 G(s)는 다음과 같다.

$$G(s) = \frac{1}{s^3 + 5s^2 + 3s + 4}$$

전달함수 G(s)는 -4.5328, $-0.2336 \pm j0.9099$에 극점을 가짐.

MATLAB 프로그램 2.1

```
 A=[-5 1 0; -3 0 1; -4 0 0];
 B=[0; 0; 1];
 C=[1 0 0];
 D=[0];
[num,den] = ss2tf(A,B,C,D)
 num =
     0    0.0000    0.0000    1.0000
 den =
   1.0000    5.0000    3.0000    4.0000
pole = roots(den)
 pole =
 -4.5328
 -0.2336 + 0.9099i
 -0.2336 - 0.9099i
```

Linearization: Skill-Assessment Exercise 3.5

The relation between spring force $f_s(t)$ and the spring displacement x_s, is $f_s(t) = 2x_s^2(t)$

The applied force is $f(t) = 10 + \delta f(t)$, where $\delta f(t)$ is small force about 10N constant force

Answer: initial state $f(0) = 10N = 2x_s^2(0)$ $\quad \therefore x_s(0) = \sqrt{5}N$

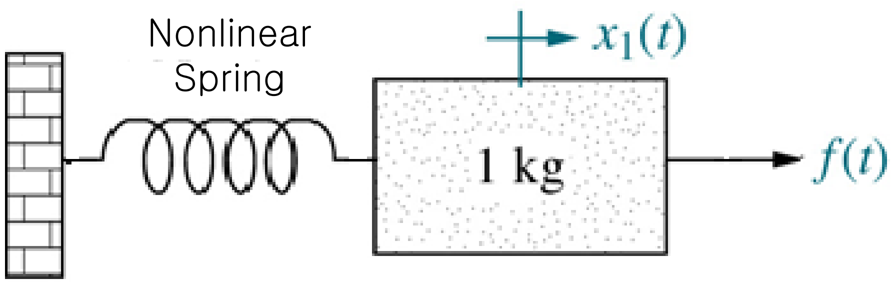


Fig. 3. 15 Nonlinear Translational mechanical system

$$f(t) - f(t)_s = M\ddot{x}_1 = M\ddot{x}_s$$

$$M\ddot{x}_s + 2x_s^2(t) = f(t)$$

$$M\ddot{x}_s + 2x_s^2(0) + 4x_s(0)x_s(\mathrm{t}) = 10 + \delta f(t)$$

$$\because f_s(t) = f_s(0) + \dot{f}_s(0)x_s + \frac{1}{2!}\ddot{f}_s(0)x_s^2 + \ldots$$

$$\approx 2x_s^2(0) + 4x_s(0)x_s(\mathrm{t})$$

$$x_1 = x_s(t) = y(t)$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{x}_s = -4x_s(0)x_s - 2x_s^{\,2}(0) + 10 + \delta f(t)$$

$$= -4x_s(0)x_s + \delta f(t) = -4\sqrt{5}x_s + \delta f(t)$$

Home Work #3 (Due date: two weeks from today)

1. Solve Problem 5 on page 147(150) in the text book. (6th edition)
2. Solve Problem 7 on page 147(150) in the text book.
3. Solve Problem 13 on page 147(151) in the text book.
4. Solve Problems 14 on page 147(151) in the text book.