Chapter 4. Time Response

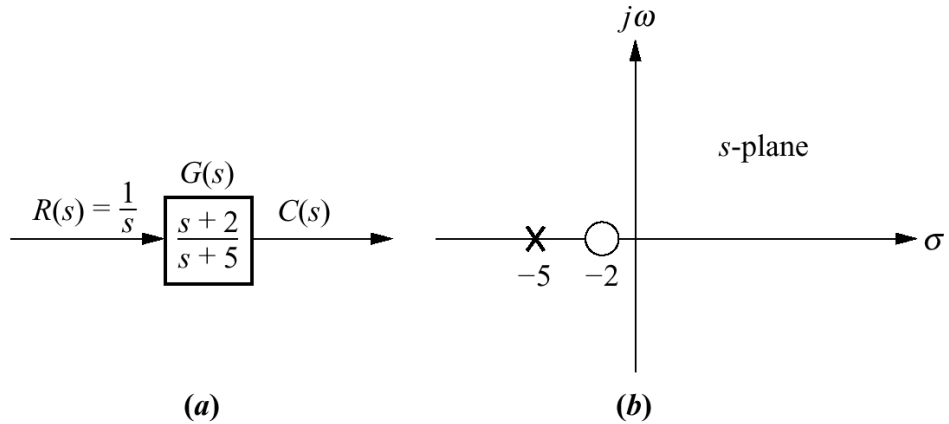
Things to know

- The time response from the transfer function
- Poles and zeros to determine the response of a control system
- The transient response
- Approximation of higher-order system
- The effects of nonlinearities on the system time response
- The time response from the state-space representation

◆ Poles and Zeros

$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{2/5}{s} + \frac{3/5}{s+5}$$

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$



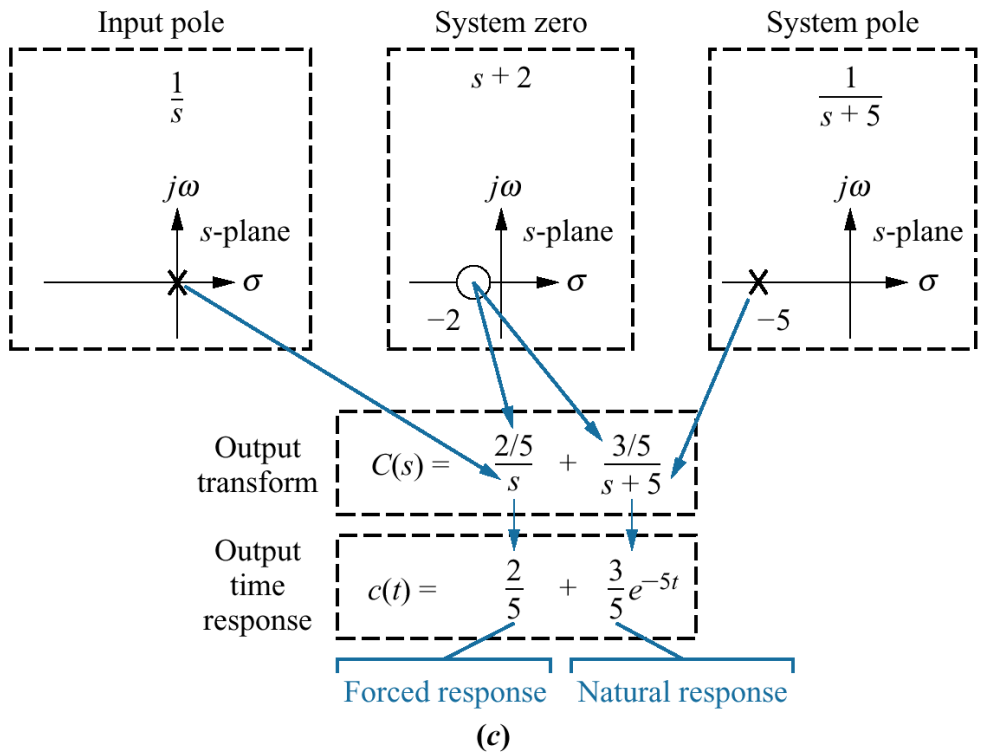
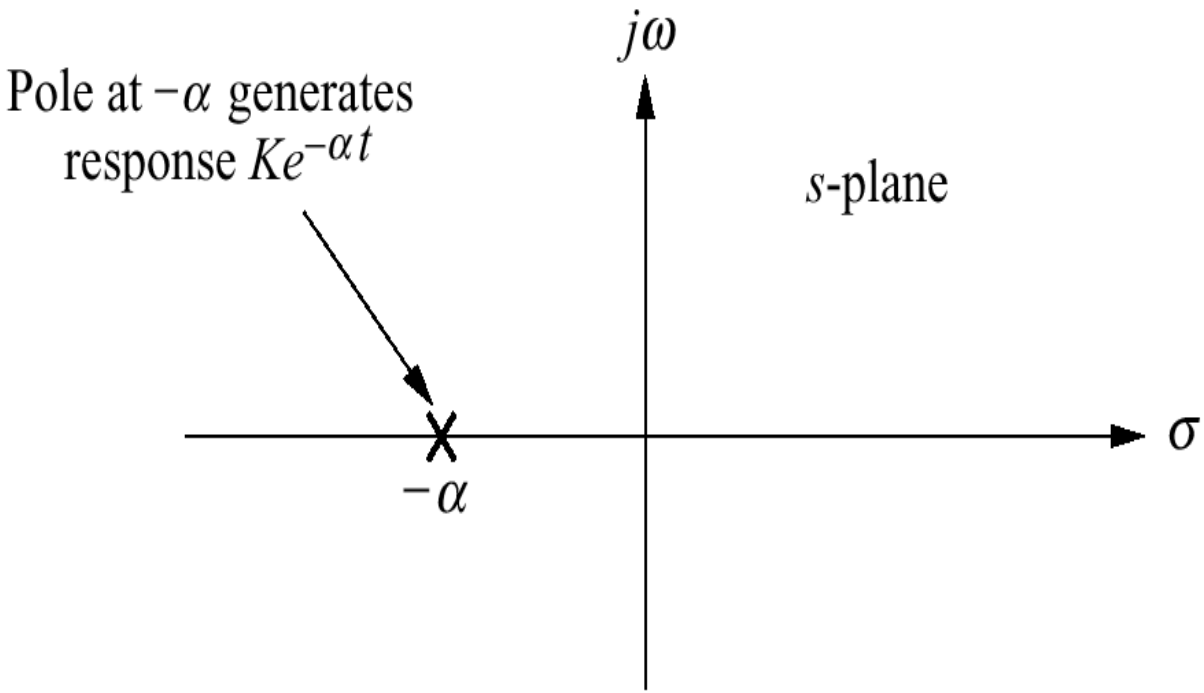


Figure 4.1
a. System showing input and output;
b. pole–zero plot of the system;
c. evolution of a system response. Follow blue arrows to see the evolution of the response component generated by the pole or zero.

1. A pole of input function generates the forced response
2. A pole of the transfer function generates the natural response
3. A pole on the real axis generates an exponential response
4. The zeros and poles generate the amplitude for both the responses

Figure 4.2
Effect of a real-axis pole upon transient response



◆ First-Order Systems

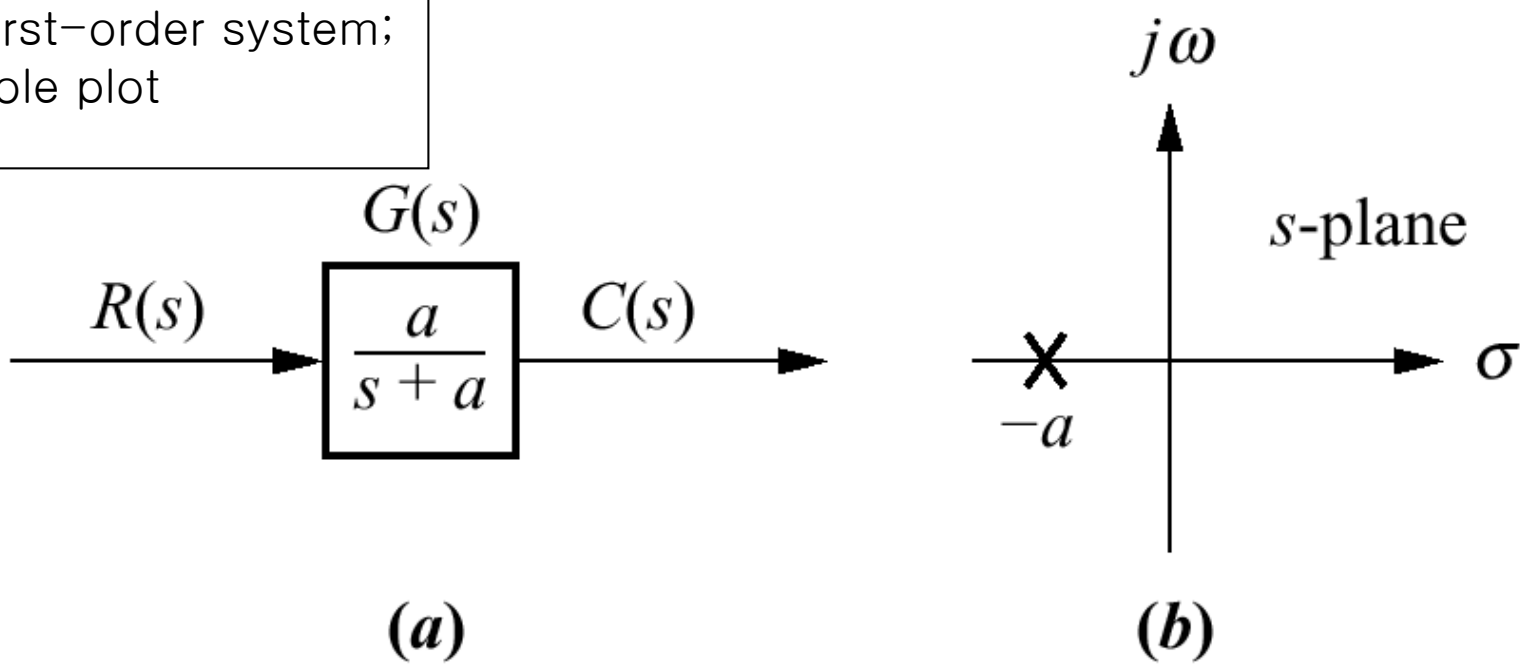
$$C(s) = R(s)G(s) = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

$$c(t) = 1 - e^{-at}$$

Time costant $t = \dfrac{1}{a}$

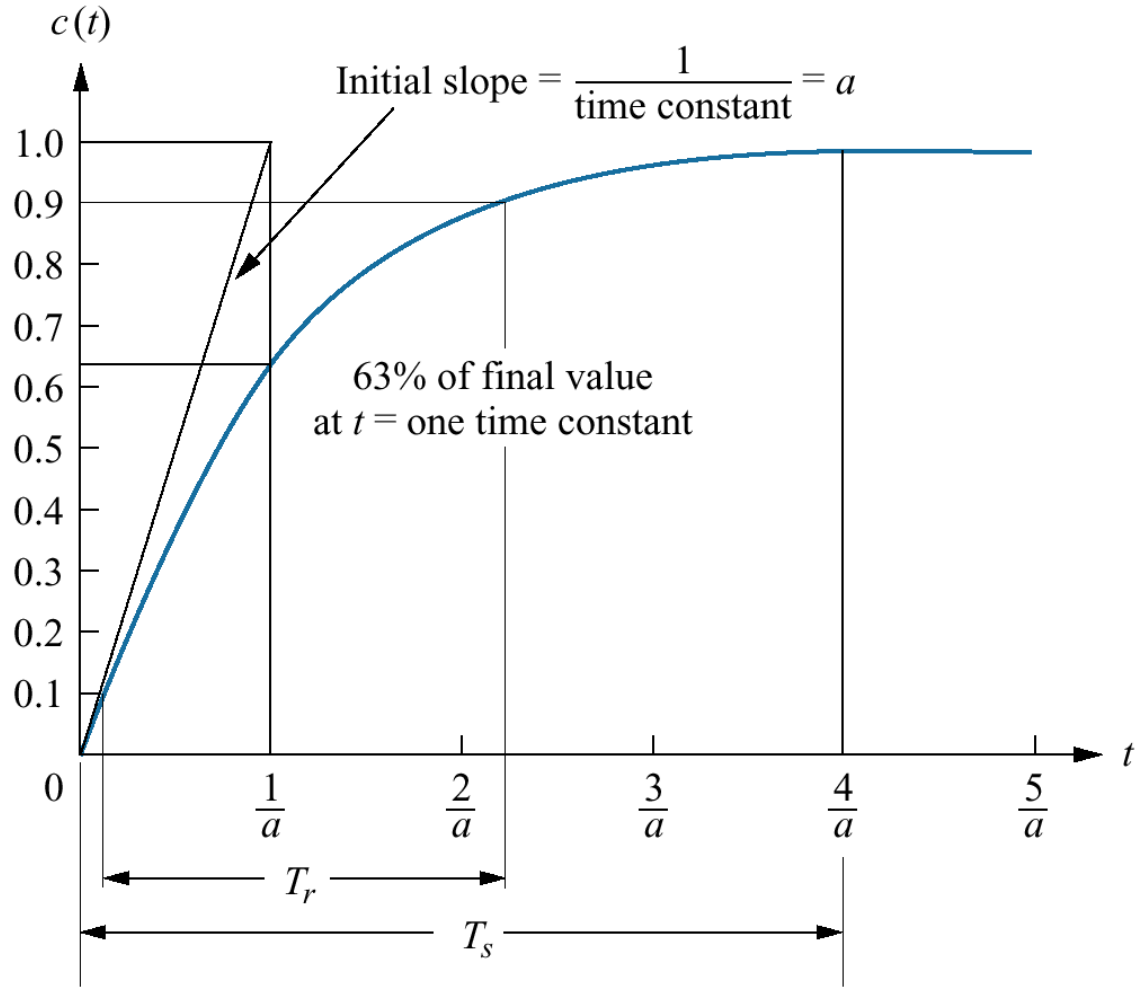
$$c(t)\big|_{t=1/a} = 1 - e^{-1} = 0.63$$

Figure 4.4
a. First-order system;
b. pole plot



◆ Response of First-Order Systems

Figure 4.5
First-order system response to a unit step



Rise Time T_r : from 0.1 to 0.9 of its final value

Settling Time T_s : within 2% of its final value

◆ Second-Order Systems

$$G(s)=\frac{y(s)}{u(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

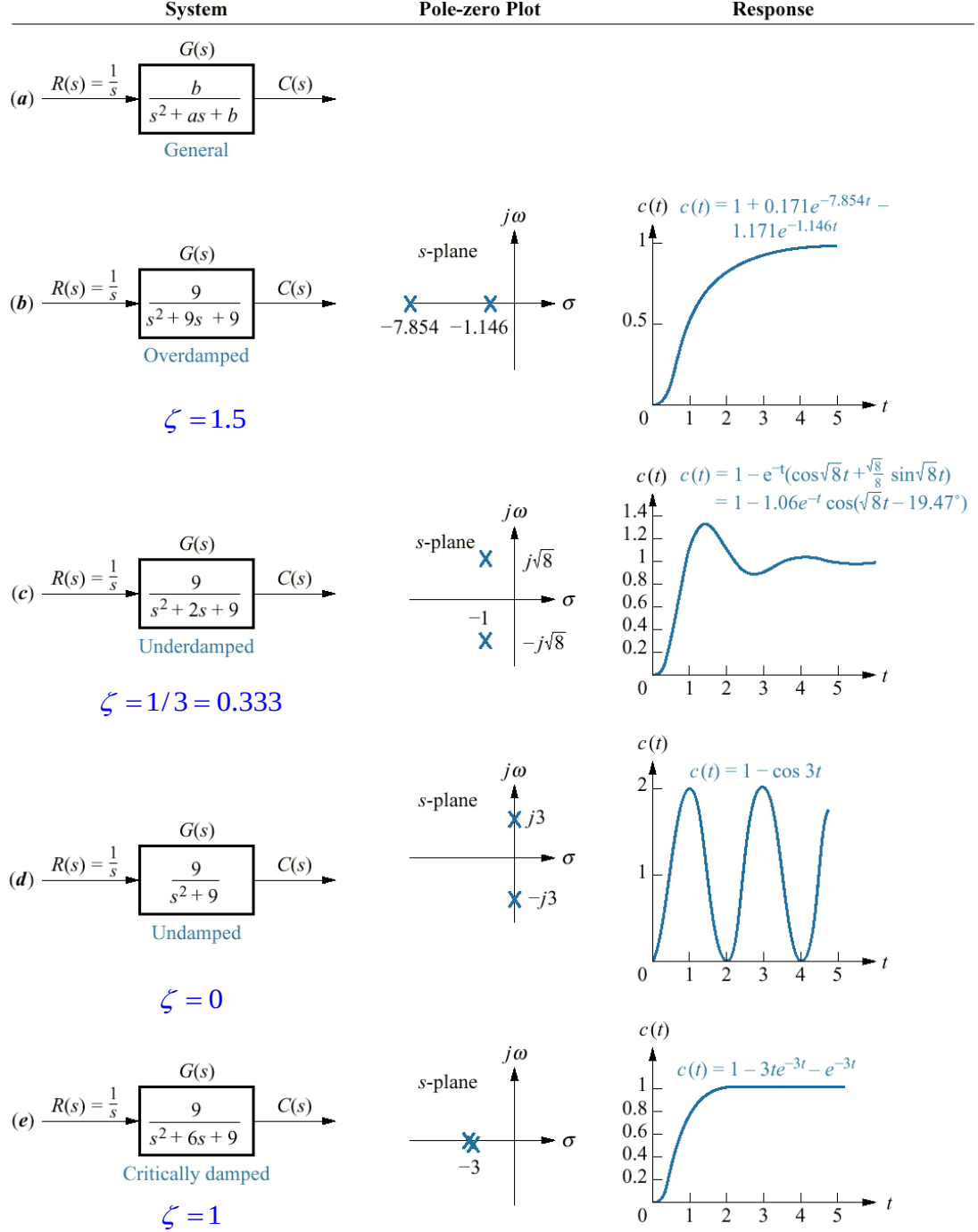


Figure 4.7
Second-order systems, pole plots, and step responses

Natural Response of G(s)

$$G(s) = \frac{y(s)}{u(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

1. Overdamped $c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$
2. Underdamped $c(t) = A e^{-\sigma_d t} \cos(\omega_d t - \phi)$
3. Undamped $c(t) = A\cos(\omega_1 t - \phi)$
4. Critically damped $c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_2 t}$

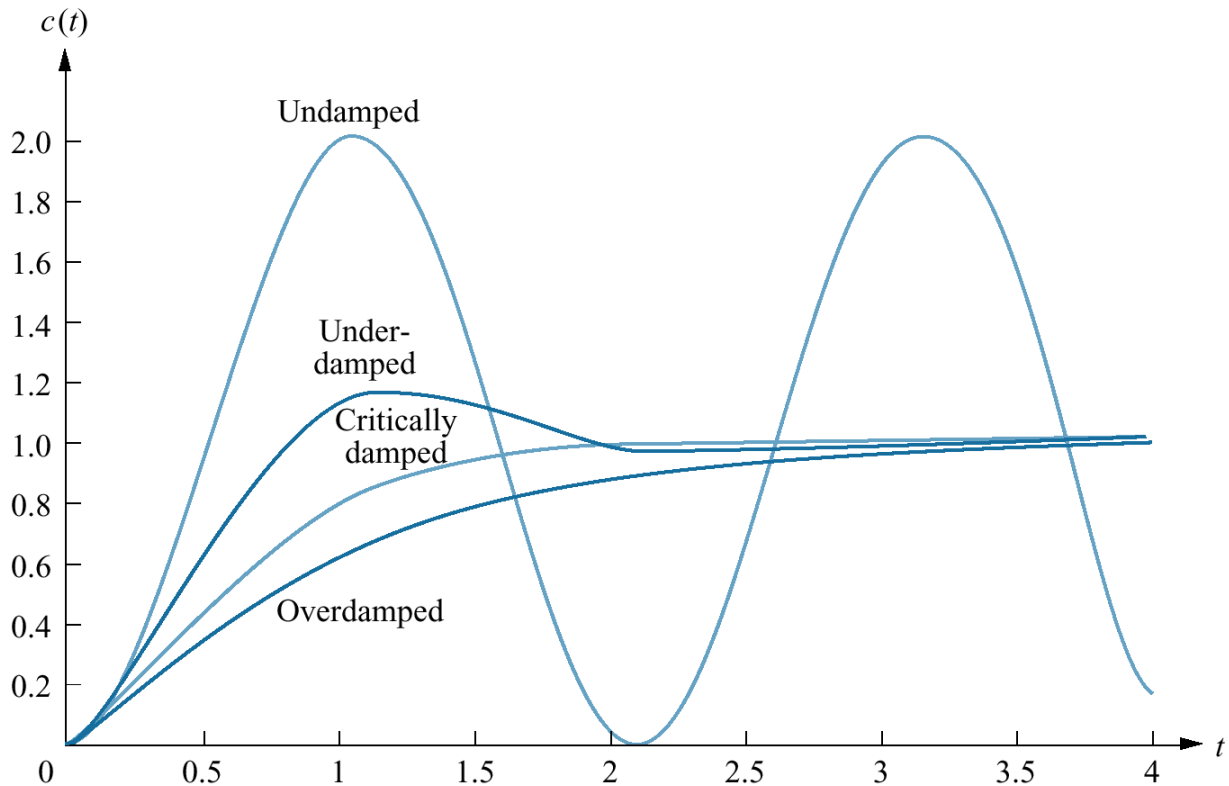


Figure 4.10
Step responses for second-order system damping cases

Norman S. Nise, California State Polytechnic University

Second-Order Step Response

Click here for Interactive Animation

Control Systems Engineering, 7th Edition

Norman S. Nise, California State Polytechnic University

Antenna (Ch. 6)

http://www.wiley.com/college/nise/0471794759/swf/AntenaChap6.swf

◆ Transient Response (과도응답)

▶ 표준 시험 입력신호:
임펄스함수, 스텝함수, 램프함수, 가속도함수, 정현파함수 등

1) 임펄스응답

$$f_\varepsilon(t) = \begin{cases} 1/\varepsilon\,, & 0 \le t \le \varepsilon \\ 0\,, & t > \varepsilon \end{cases}$$

$$\int_0^\infty \delta(\tau)d\tau = 1$$

$$\int_0^\infty \delta(t-\tau)g(\tau)d\tau = \int_0^\infty g(t-\tau)\delta(\tau)d\tau = g(t)$$

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

$$y(t) = \int_0^t g(t-\tau)\delta(\tau)d\tau = g(t)$$

2) 스텝응답

동적 시스템의 성능시험에 주로 사용되고 있다.

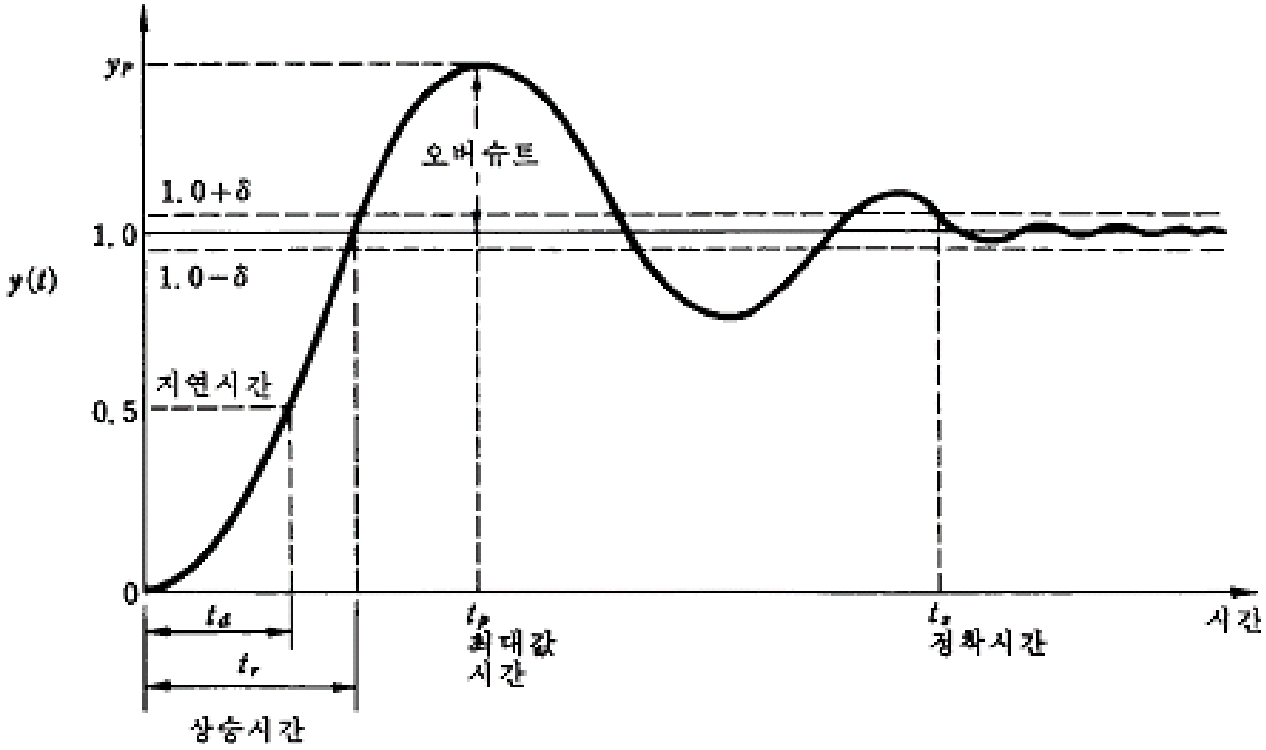
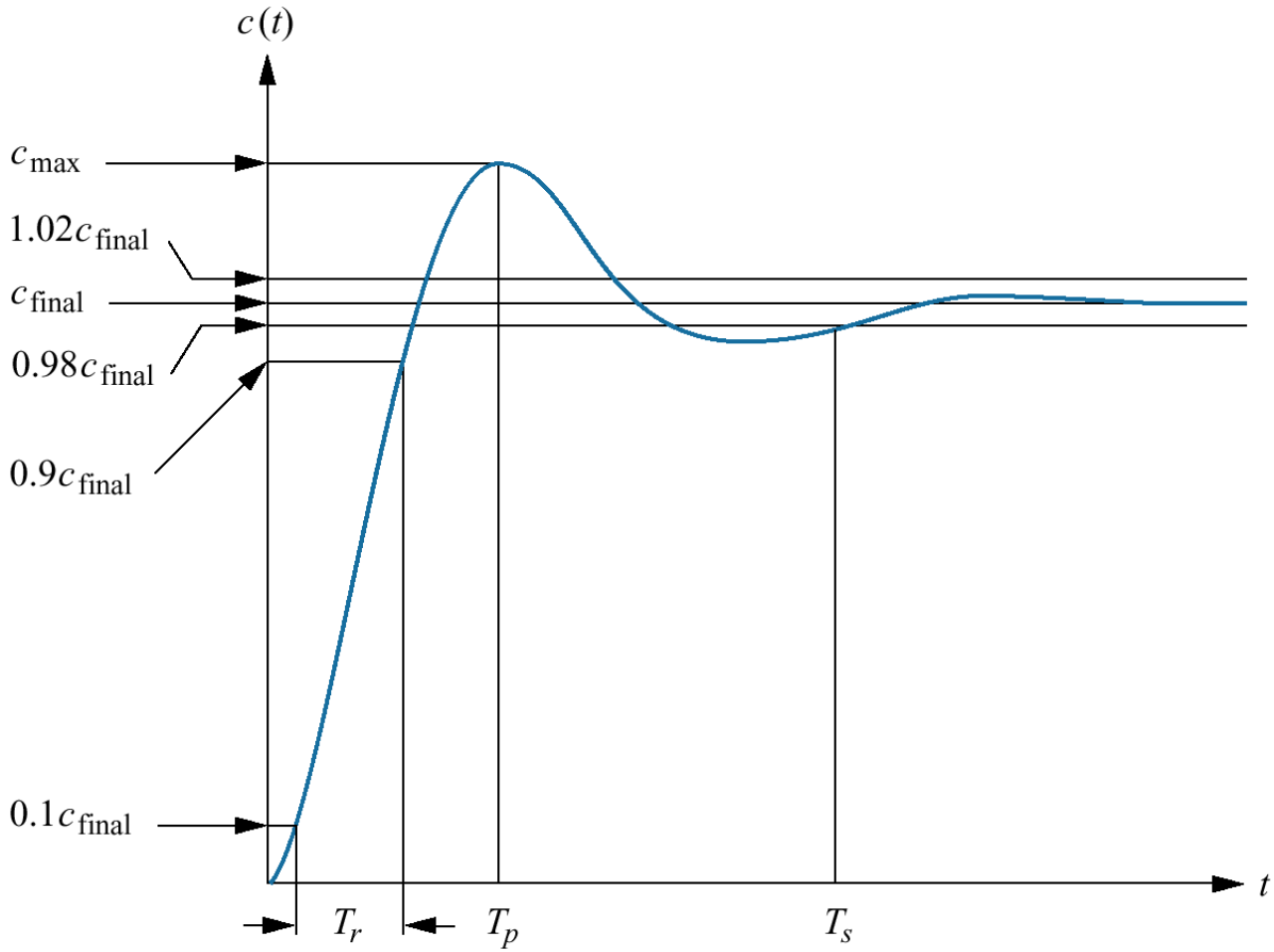


그림 3.6 제어시스템의 전형적인 단위스텝응답

Figure 4.14
Second-order underdamped response specifications



1. Rise time T_r : from 0.1 to 0.9 of its final value
2. Peak time T_p : to reach the first peak
3. Percent overshoot : $P.O. = \dfrac{c_{max} - c_{ss}}{c_{ss}} \times 100\%$
4. Settling time T_s : within 2% of the steady state value

▶ 유한 영점이 없는 단순 2차 시스템의 해석

단순 2차 시스템의 일반형은 다음과 같다.

$$y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} u(s) \qquad (3.49)$$

단위스텝입력 (u(s) = 1/s)을 가했을 때의 출력 y(s)는 다음과 같다

$$y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \qquad (c_1 s + c_2)_{s=p_1} = \lim_{s \to p_1}\left[\frac{N(s)}{D(s)}(s-p_1)(s-p_2)\right] \qquad (3.50)$$

그리고, 위 식을 역 Laplace 변환하여 단위스텝응답 y(t)를 구하면

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta) \qquad (3.51)$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2}, \quad \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

시정수 T: 감쇠 정도를 나타내는 지수함수 $e^{-\zeta\omega_n t}$ 가 e^{-1} 이 되는 시간,

즉, T = 1/ζω$_n$으로 정의된다.

Proof) $\lim_{s\to p_1}\left[\frac{N(s)}{D(s)}(s-p_1)(s-p_2)\right]=\lim_{s\to p_1}\left(\frac{c_1}{s}+\frac{c_2 s+c_3}{(s-p_1)(s-p_2)}\right)(s-p_1)(s-p_2)=(c_2 s+c_3)_{s=p_1}$

$$y(s)=\frac{1}{s}\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{1}{s}\frac{\omega_n^2}{(s-p_1)(s-p_2)}=\frac{c_1}{s}+\frac{c_2 s+c_3}{(s-p_1)(s-p_2)}$$

단, $p_1=-\sigma+j\omega_d$, $p_2=-\sigma-j\omega_d$, (여기서 $\sigma=\zeta\omega_n$, $\omega_d=\omega_n\sqrt{1-\zeta^2}$)

$$c_1=sy(s)\big|_{s=0}=1,\quad (c_2 s+c_3)\big|_{s=p_1}=\frac{1}{s}\frac{\omega_n^2}{(s-p_1)(s-p_2)}(s-p_1)(s-p_2)\Big|_{s=p_1}$$

$$y(s)=\frac{1}{s}-\frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}=\frac{1}{s}-\frac{(s+\zeta\omega_n)+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$\because)\ c_2(-\sigma+j\omega_d)+c_3=\frac{\omega_n^2}{(-\sigma+j\omega_d)}=\frac{\omega_n^2(\sigma+j\omega_d)}{(-\sigma+j\omega_d)(\sigma+j\omega_d)}=-(\sigma+j\omega_d)$$

$$\therefore c_2=-1,\quad c_3=-2\zeta\omega_n$$

$$y(s) = \frac{1}{s} - \frac{(s+\zeta\omega_n)+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$\therefore\ y(s) = \frac{1}{s} - \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)} - \frac{\zeta}{\sqrt{(1-\zeta^2)}}\frac{\omega_n\sqrt{(1-\zeta^2)}}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$\therefore\ y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{(1-\zeta^2)}}\sin\omega_d t\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\left(\sqrt{(1-\zeta^2)}\cos\omega_d t + \zeta\sin\omega_d t\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}}\left(\sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t\right)$$

$$\left.\begin{aligned} &\therefore\ y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta) \\ &\omega_d = \omega_n\sqrt{1-\zeta^2}\ , \quad \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} \end{aligned}\right)$$

또는

$$\left(\begin{aligned} &y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \phi) \\ &\phi = \sin^{-1}\zeta \ \ or \ \ \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}} \end{aligned}\right.$$

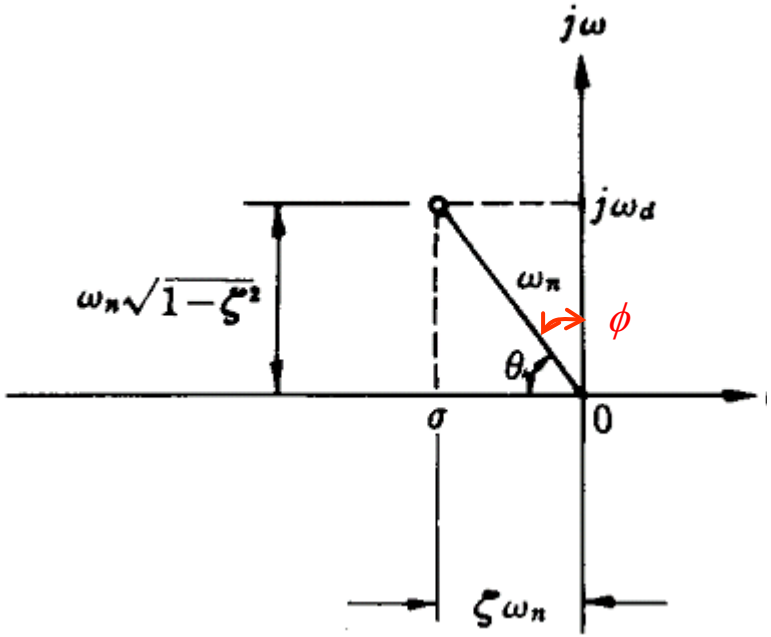


그림 3.7 감쇠고유주파수
ω_d 와 각 θ 의 표시

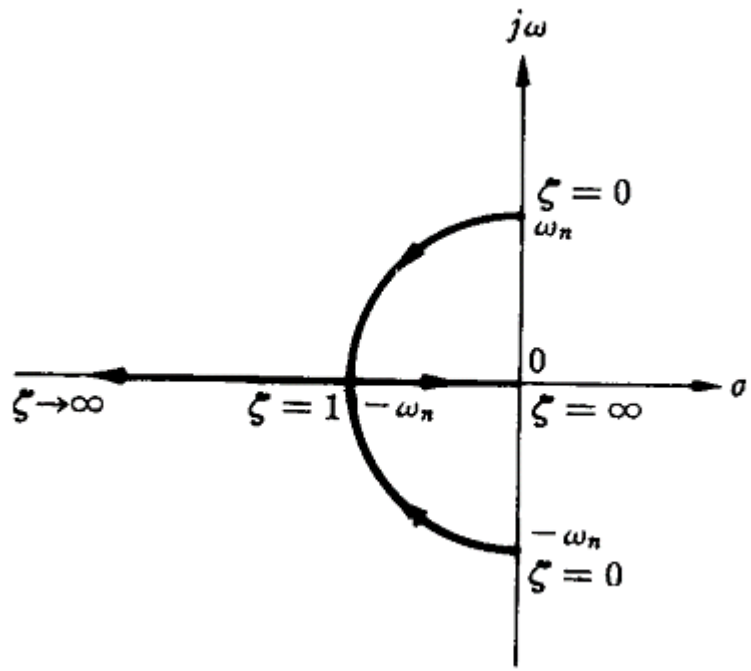
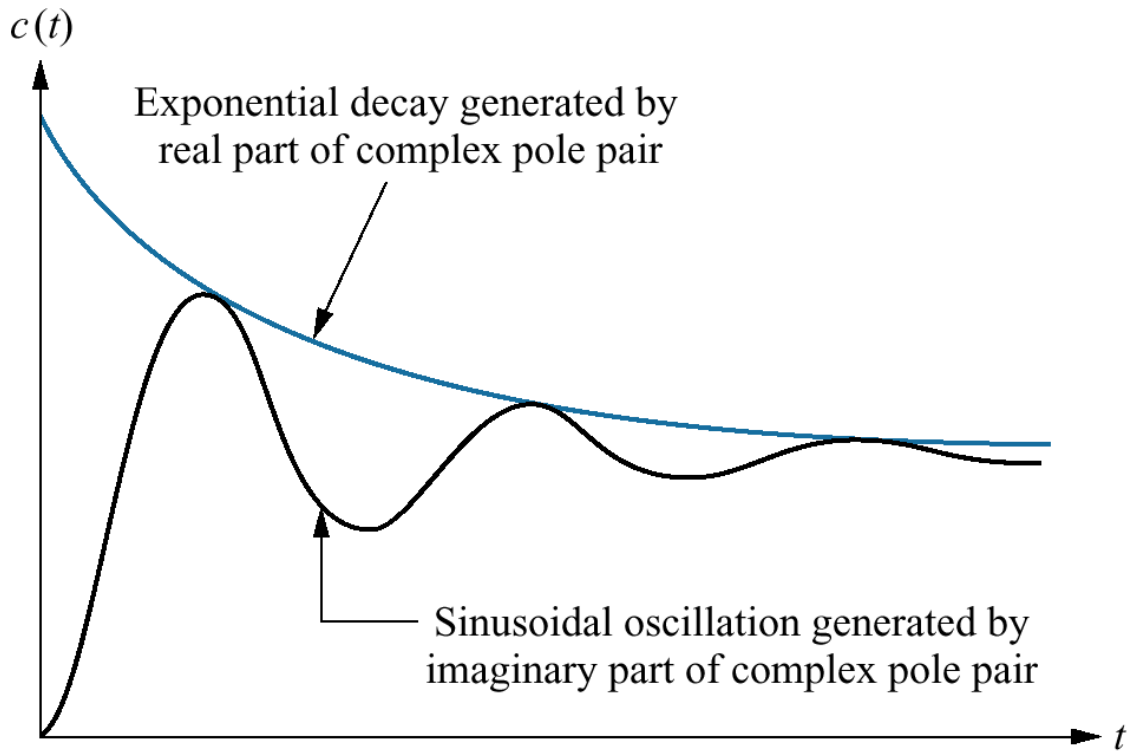


그림 3.8 감쇠비 ζ 값에 따른
단순 2차 시스템의 극점 변화

Figure 4.8
Second-order step response components generated by complex poles



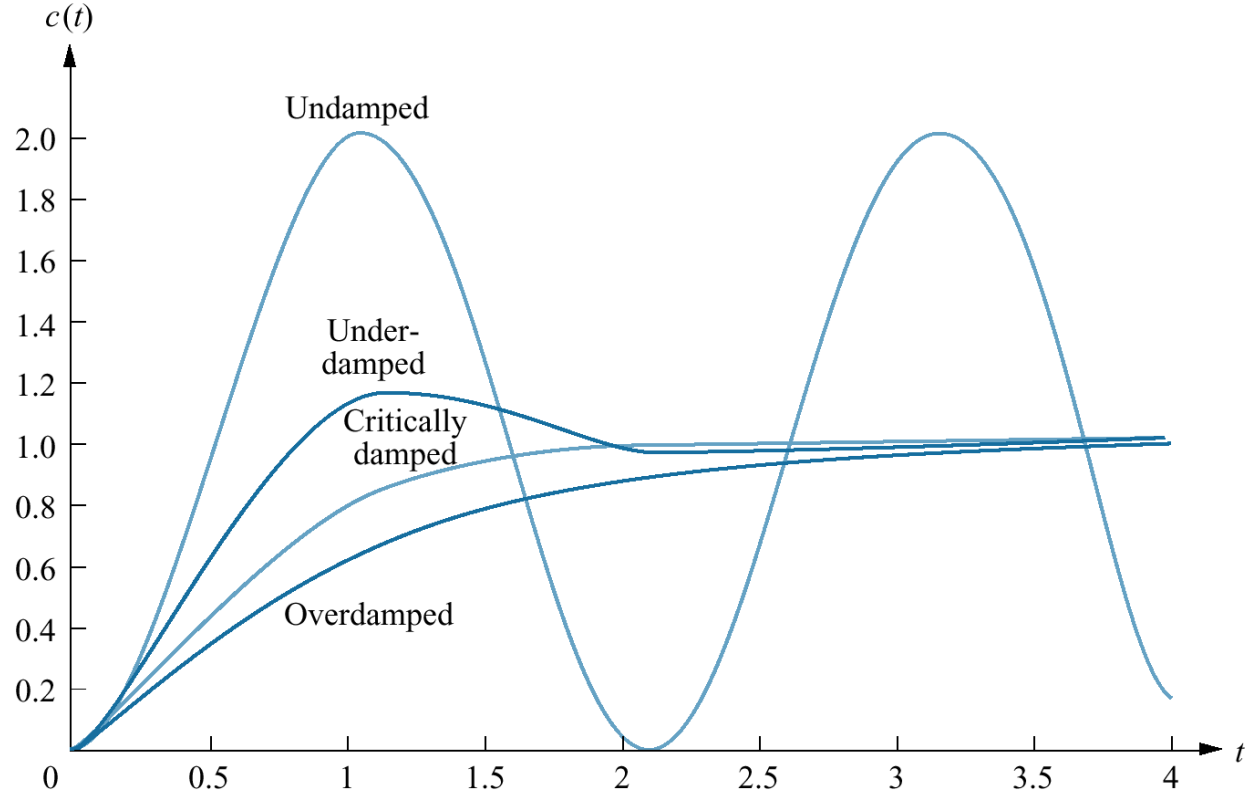
$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}\,, \quad \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \phi)$$

$$\phi = \sin^{-1}\zeta$$

Figure 4.10
Step responses for second-order system damping cases



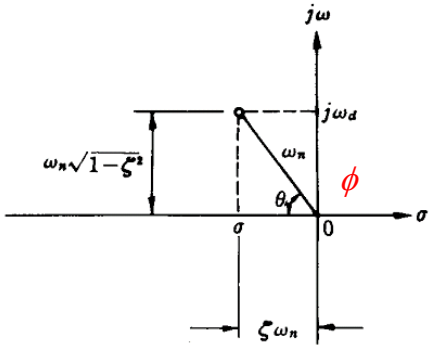
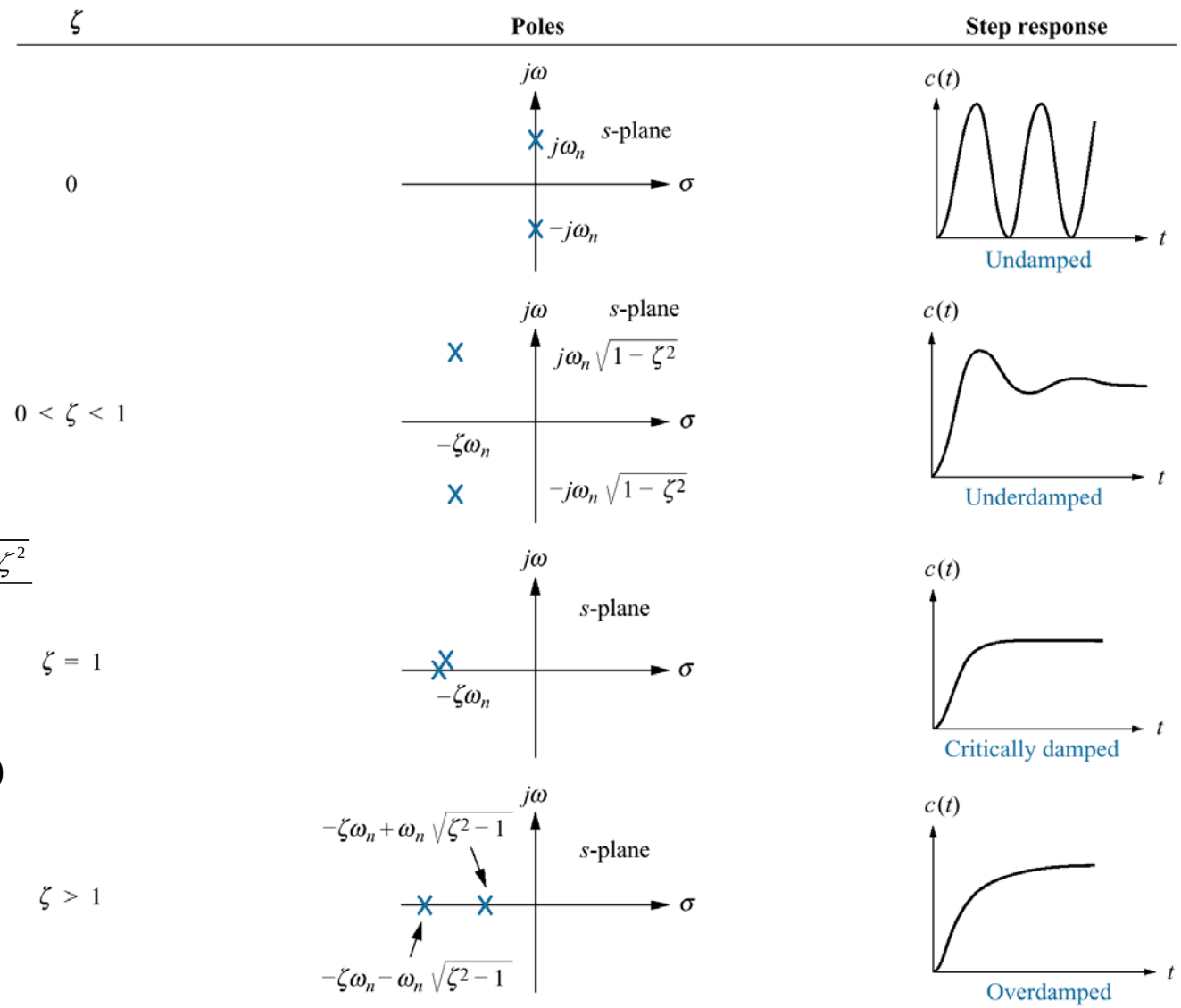


Figure 4.11
Second-order response as a function of damping ratio



$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}\ , \quad \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}u(s)$$

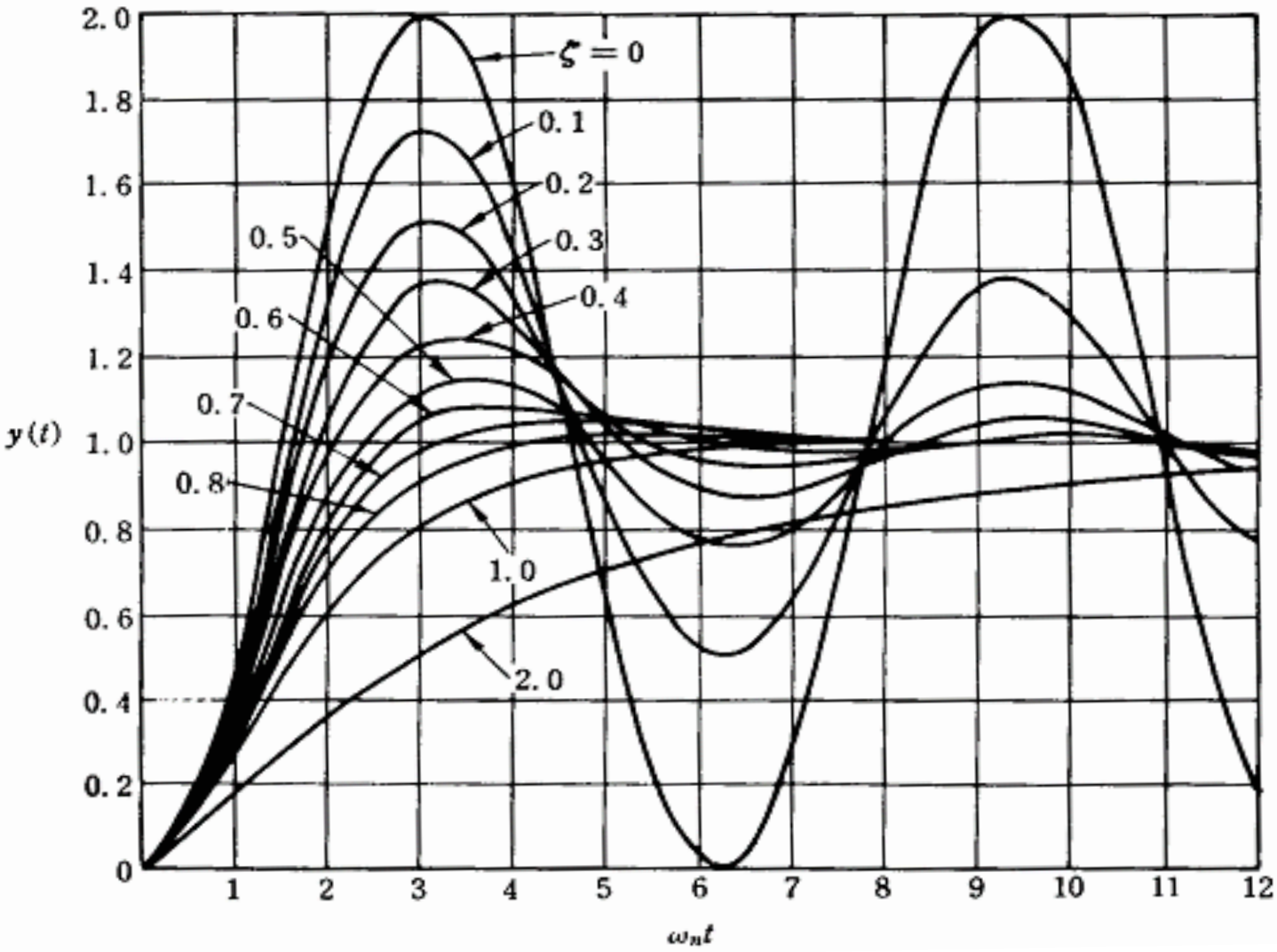


그림 3.9 ζ값에 따른 단순 2차 시스템의 단위스텝응답

Norman S. Nise, California State Polytechnic University

Second-Order Step Response

Click here for Interactive Animation

▶ 단순 2차 시스템의 과도응답에 관한 사양 계산

1) Peak time T_p

$$y_p = 1 + e^{-(\zeta/\sqrt{1-\zeta^2})\pi} \quad , T_p = \frac{\pi}{\omega_d} \quad \Leftarrow \quad \left.\frac{dy}{dt}\right|_{t=T_p} = 0$$

2) Rise time T_r

$$\omega_d T_r + \theta = \pi \quad \text{또는} \quad T_r = \frac{\pi - \theta}{\omega_d}$$

3) Settling time T_s :

2% 정착시간 T_s는 감쇠비 ζ값에 거의 무관하게

$$T_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n}$$

5% 정착시간 t_s는 시스템의 시정수 T의 3배가 된다.

$$T_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n}$$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}\,, \quad \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

▶ 단순 2차 시스템의 과도응답에 관한 사양 계산

1) Peak time

$$\left.\frac{dy}{dt}\right|_{t=T_p} = \frac{\zeta\omega_n e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d T_p + \theta) - \frac{\omega_n\sqrt{1-\zeta^2}\,e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\cos(\omega_d T_p + \theta)$$

$$= \frac{\omega_n e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\left(\cos\theta\sin(\omega_d T_p + \theta) - \sin\theta\cos(\omega_d T_p + \theta)\right)$$

$$= \frac{\omega_n e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\sin\omega_d T_p = 0$$

$$\therefore\ \omega_d T_p = 0, \pi, 2\pi, 3\pi, \cdots$$

=> 최대값 시간 T_p는 첫번째 오버슈트의 시간 $\therefore \omega_d T_p = \pi, \quad T_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

$$\therefore y_p = 1 - \frac{e^{-(\zeta/\sqrt{1-\zeta^2})\pi}}{\sqrt{1-\zeta^2}}\underbrace{\sin(\pi+\theta)}_{(=-\sqrt{1-\zeta^2})} = 1 + e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}\ , \quad \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

2) Rise time T_r

$$y(T_r) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d T_r + \theta) = 1$$

$$\therefore \omega_d T_r + \theta = \pi, \quad T_r = \frac{\pi - \theta}{\omega_d}$$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \phi)$$

$$\phi = \sin^{-1}\zeta$$

3) Settling time T_s

2% 정착시간 t_s 는 감쇠비 ζ값에 거의 무관하게 시정수 $T = \dfrac{1}{\zeta\omega_n}$ 의 4배

$$T_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n}$$

$$If\ \cos(\omega_n\sqrt{1-\zeta^2}t - \phi) = 1, \quad \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} = 0.02 \quad \Rightarrow T_S = \frac{-\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n}$$

5% 정착시간 t_s는 시스템의 시정수 T의 3배가 된다.

$$T_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n}$$

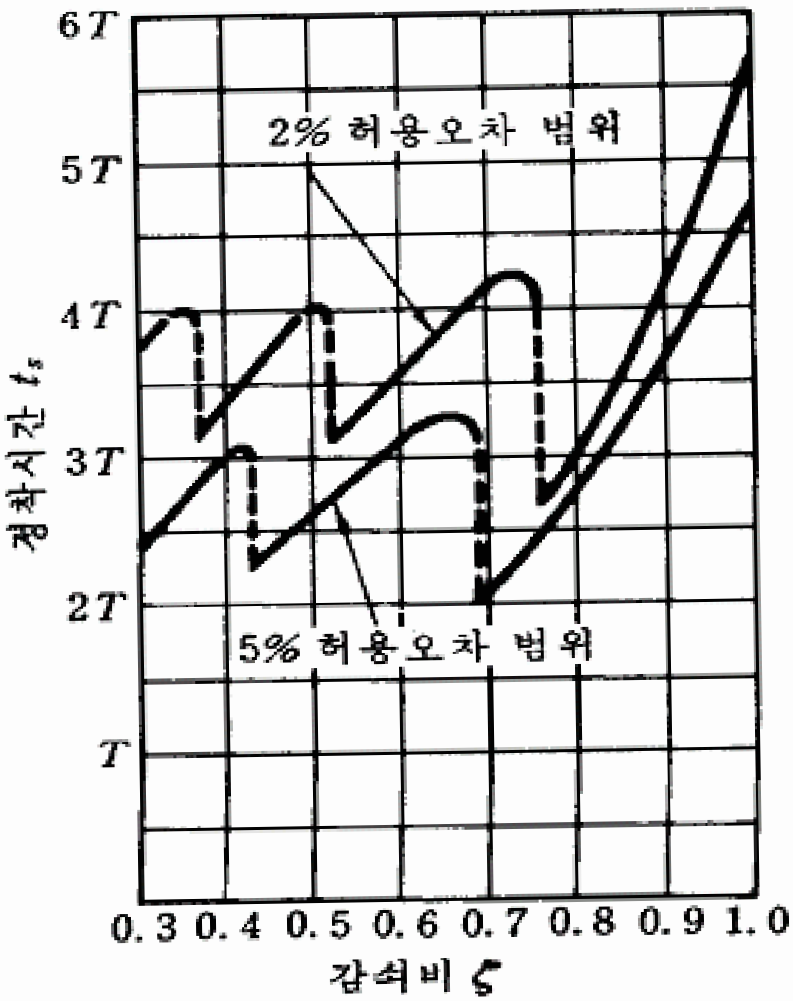


그림 3.10 감쇠비 ζ값에 따른 정착시간 t_s

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)$$

4) Percentage Overshoot %OS

$$t_p = \pi / \omega_d = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\%OS = \frac{y_{\max} - y_{final}}{y_{final}} \times 100$$

$$y_{\max} = y(T_p) = 1 - \frac{e^{-(\varsigma\pi/\sqrt{1-\varsigma^2})}}{\sqrt{1-\zeta^2}}\underbrace{\sin(\pi+\theta)}_{(=-\sqrt{1-\zeta^2})}$$

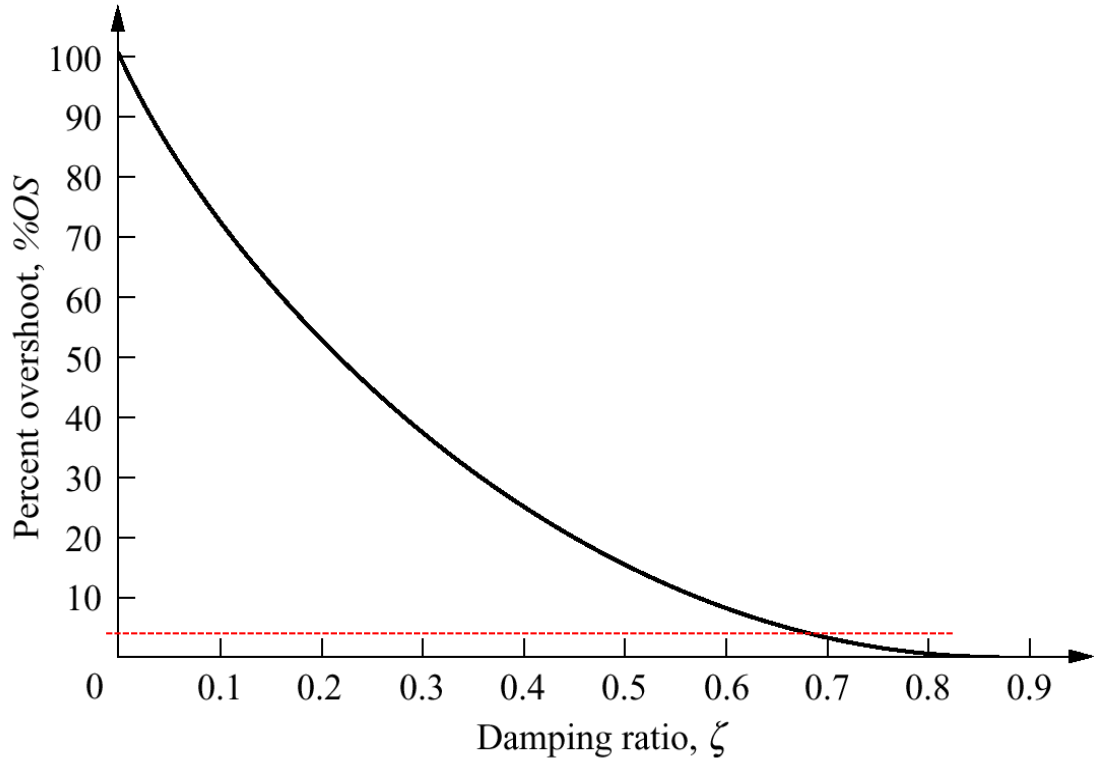
$$= 1 + e^{-(\varsigma\pi/\sqrt{1-\varsigma^2})}$$

$$\%OS = (y_{\max} - 1) \times 100 = e^{-(\varsigma\pi/\sqrt{1-\varsigma^2})} \times 100$$

$$\frac{\varsigma\pi}{\sqrt{1-\varsigma^2}} = -\ln\frac{\%OS}{100} = A, \qquad \varsigma^2(\pi^2 + A^2) = A^2$$

$$\therefore\ \varsigma = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

Figure 4.15
Percent overshoot vs. damping ratio

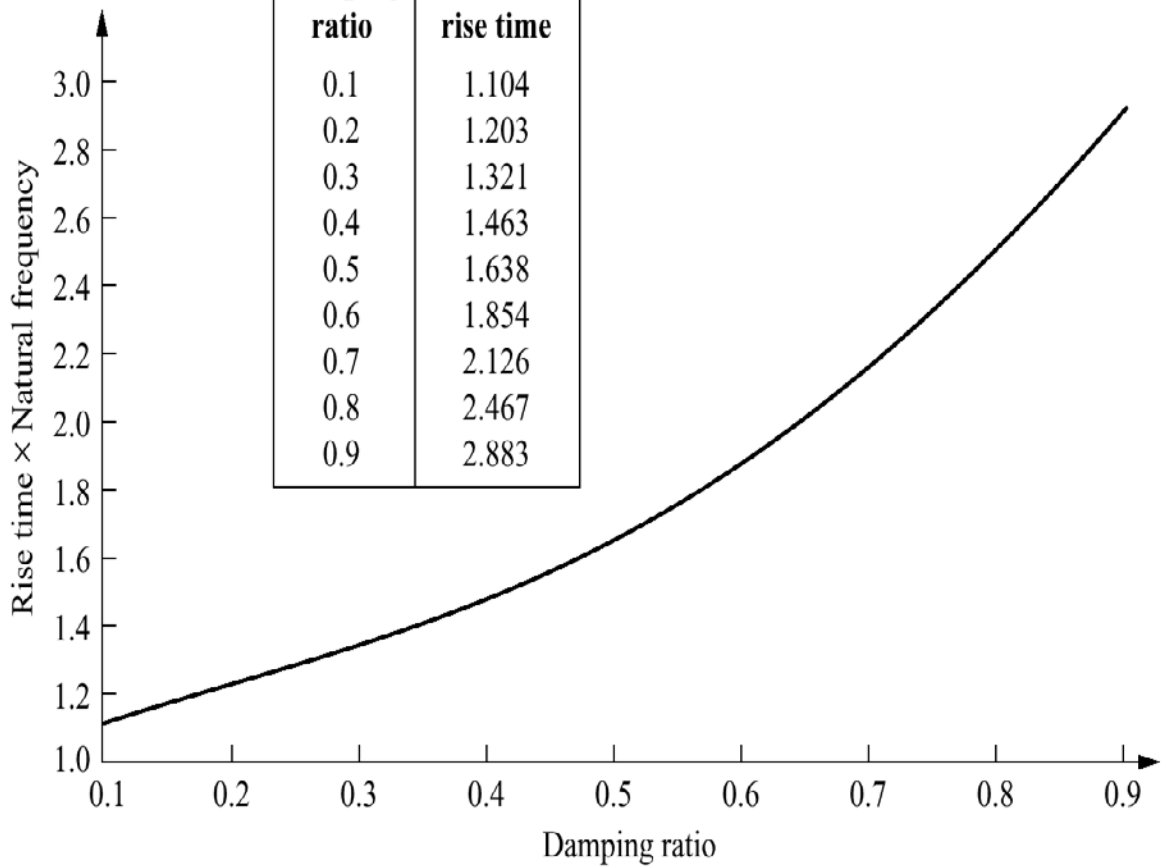


$$\%OS = e^{-(\varsigma\pi/\sqrt{1-\varsigma^2})} \times 100$$

$$\varsigma = 0.7 \quad \Rightarrow \quad \%OS = 4.6\%$$

Figure 4.16
Normalized rise time vs. damping ratio for a second-order underdamped response

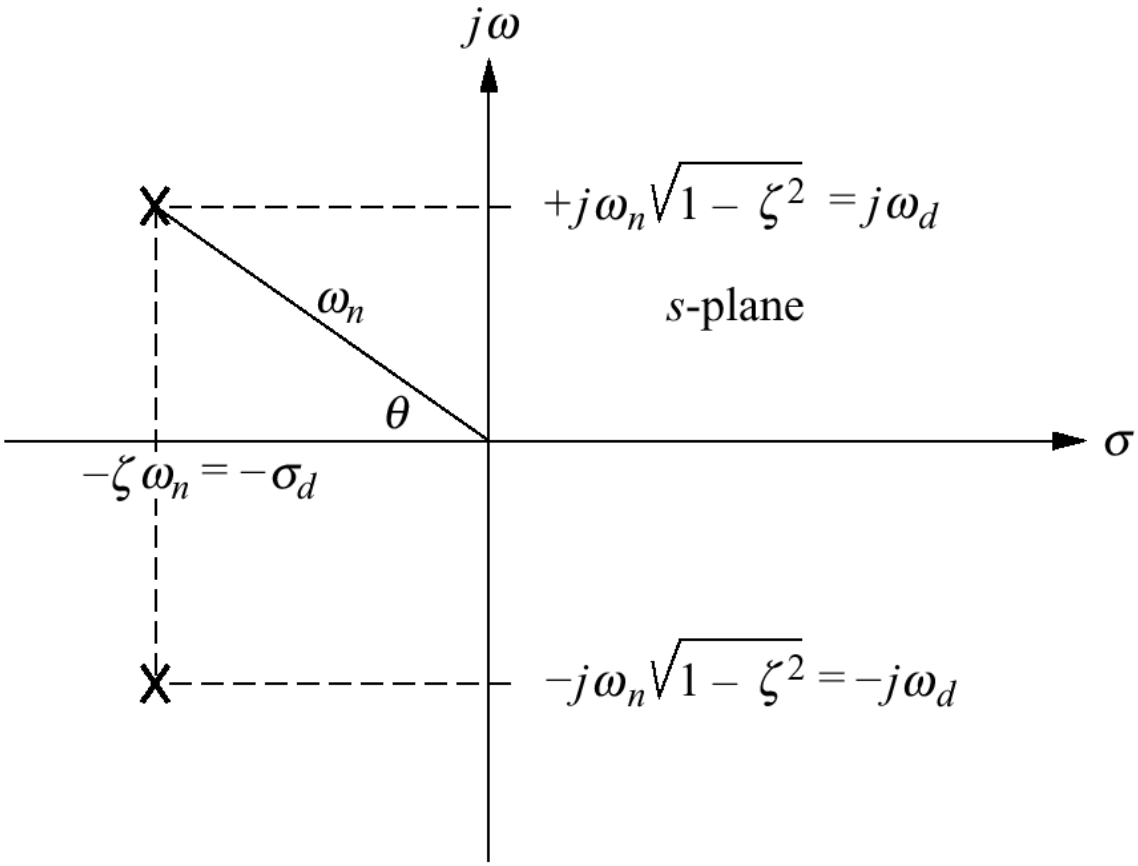
$$t_r = \frac{\pi - \theta}{\omega_d}$$



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

Figure 4.17
Pole plot for an underdamped second-order system

Finding T_P, %OS, $T_{s,}$ and T_r from a transfer function



$$G(s) = \frac{100}{s^2 + 15s + 100}$$

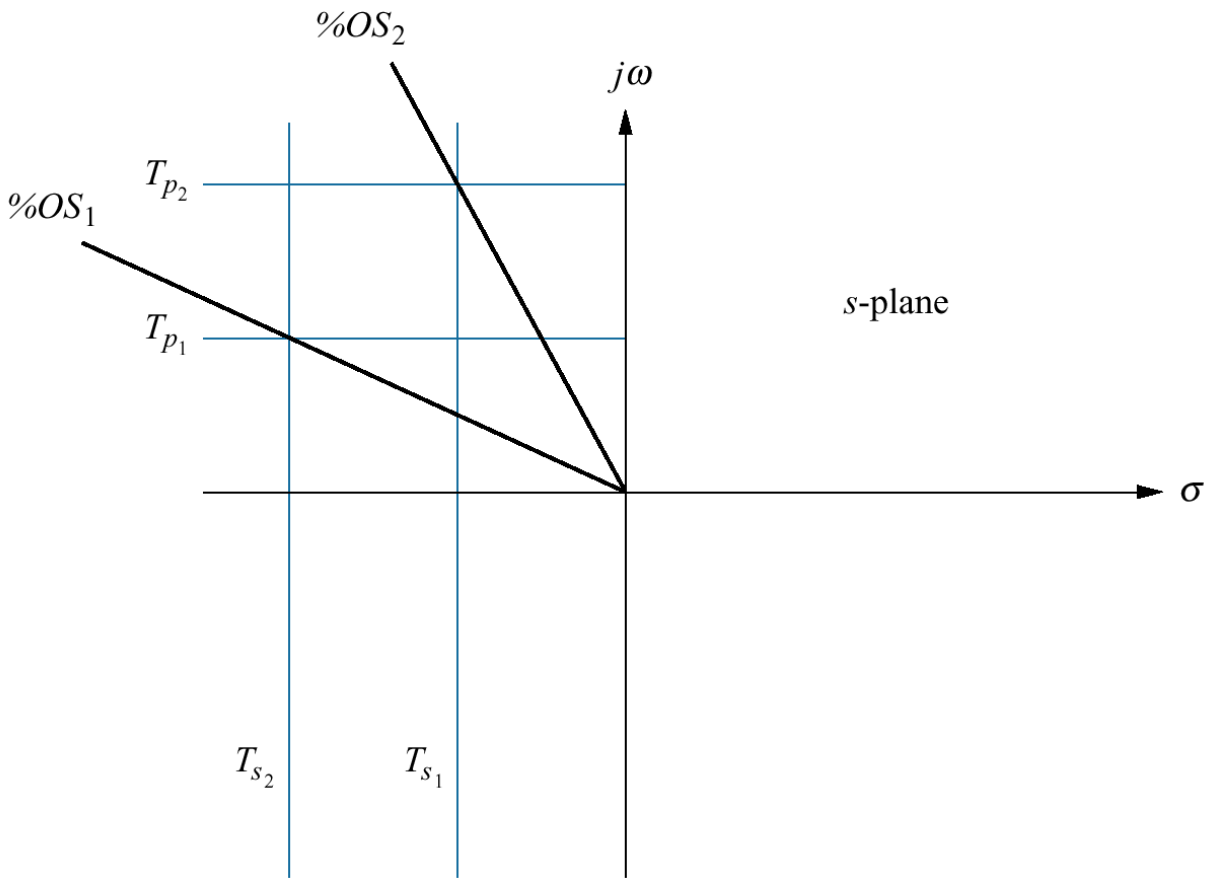
$$\omega_n = 10,\ \zeta = 0.75$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.475$$

$$\%\text{OS} = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100 = 2.838$$

$$T_s = \frac{4}{\zeta\omega_n} = 0.533\ ,\quad T_r = \frac{\pi-\theta}{\omega_d} = 0.23,\quad \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

Figure 4.18
Lines of constant peak time, T_p, settling time, T_s, and percent overshoot, %OS
Note: $T_{s_2} < T_{s_1}$; $T_{p_2} < T_{p_1}$; $\%OS_1 < \%OS_2$



$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.475$$

==> 시스템의 감쇠비 ζ는 0.4와 0.8 사이에 있는 것이 바람직

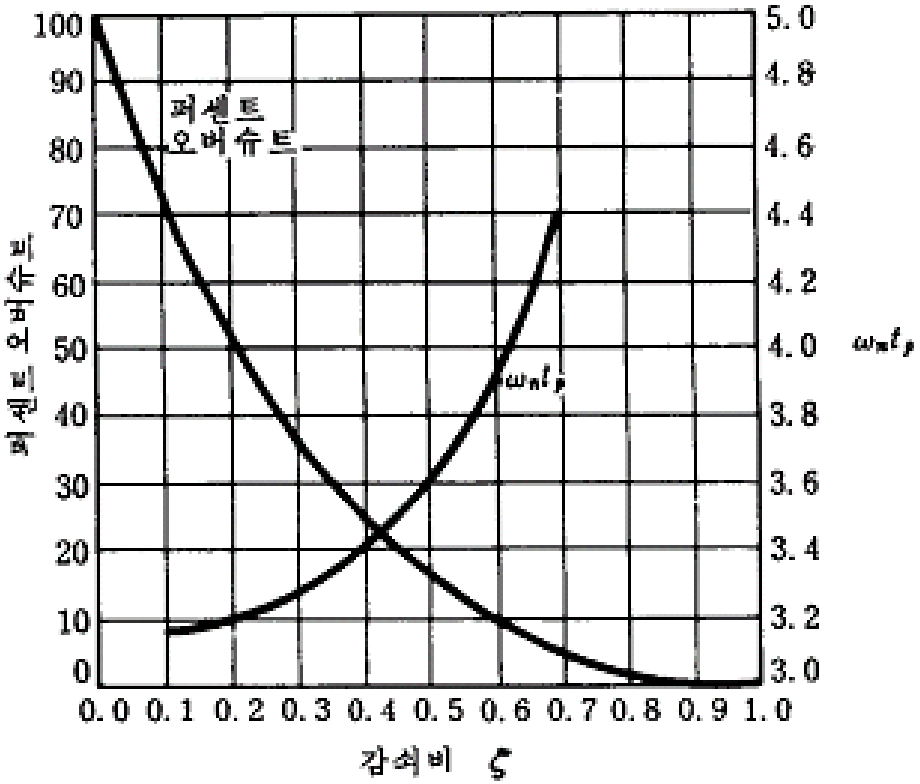


그림 3.11 단순 2차 시스템에서 감쇠비 ζ값에 따른
퍼센트 오버슈터와 최대값 시간

4.8 System Response with Zeros

$$T(s) = \frac{C(s)}{U(s)} = \frac{1}{s^2 + 2\varsigma\omega_n s + \omega_n^{\ 2}}$$

인 시스템에 영점항 $(s+a)$ 를 추가할 경우의 출력은

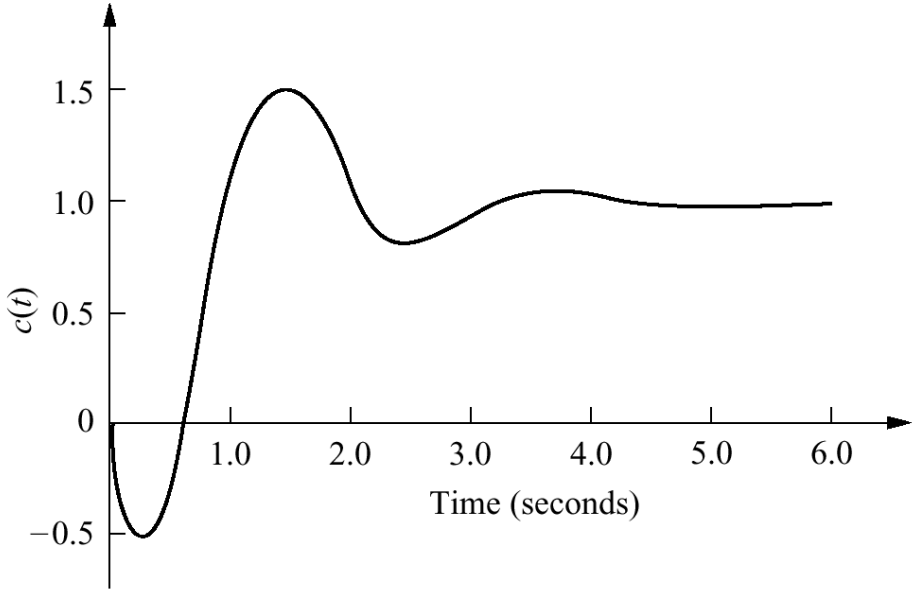
$$(s+a)T(s) = \frac{C(s)}{U(s)}(s+a) = \frac{s+a}{s^2 + 2\varsigma\omega_n s + \omega_n^{\ 2}}$$ 에서

$$(s+a)C(s) = sC(s) + aC(s)$$

즉 출력 $c(t)$ => $\dot{c}(t) + ac(t)$

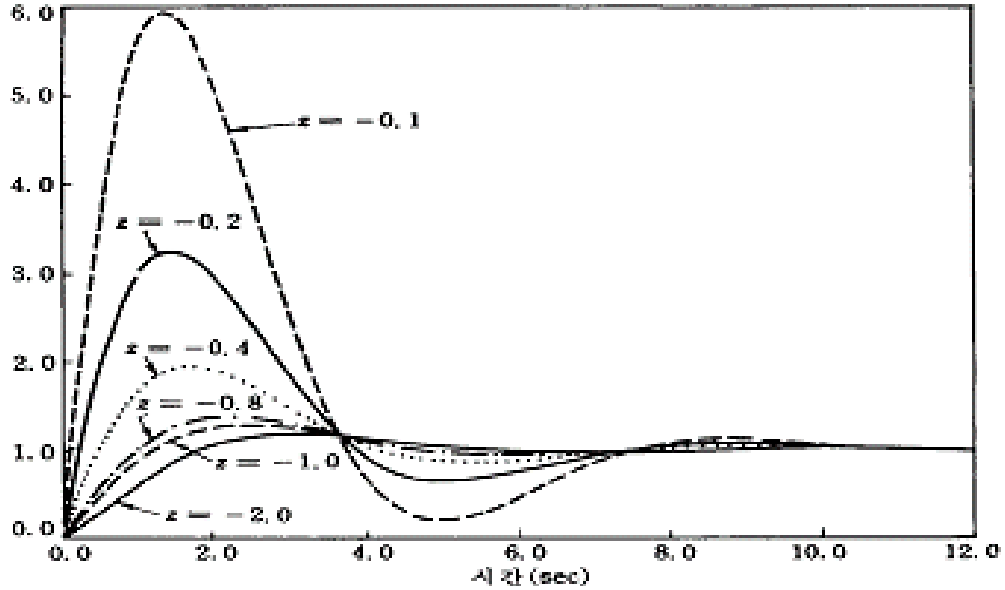
$s = -a > 0$ 일 경우

nonminimum phase system



● 시스템의 영점의 위치에 따른 과도응답 특성

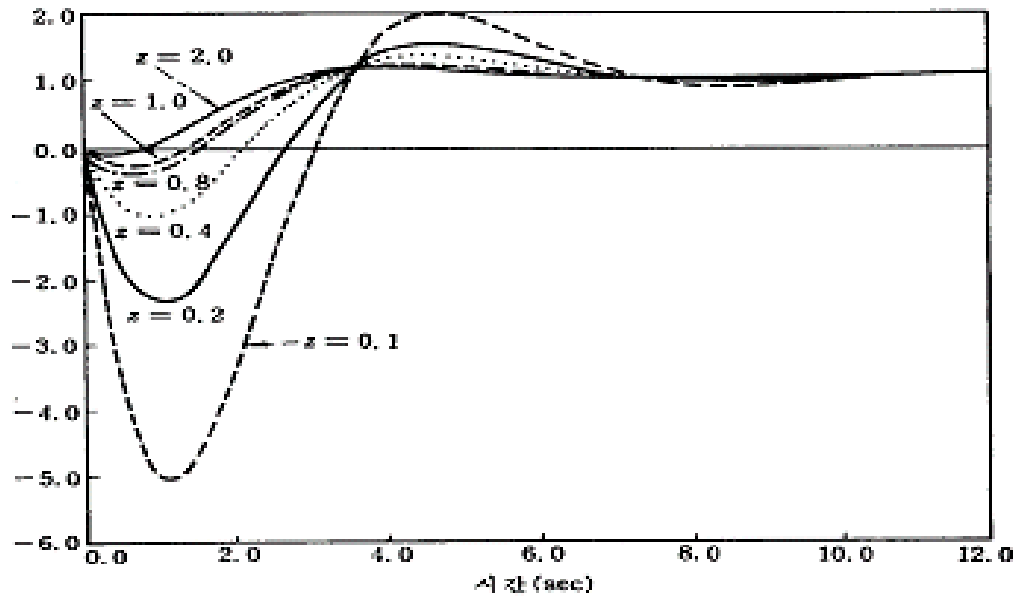
$$G(s) = \frac{\frac{1}{z}(z-s)}{s^2+s+1}$$



(a) 최소위상 시스템의 경우

영점이 허수축에 가까울 수록 영향 큼

$$(sG(s) = \frac{\frac{1}{z}(z-s)}{s^2+s+1}$$



(b) 비최소위상 시스템의 경우

▶ 시스템의 대표 극점(dominant pole)

시스템 극점들 중에서 과도응답이 가장 오래 지속되어 시스템 응답에 지배적인 역할을 하는 극점이다.
=> s-평면상에서 가장 허수축에 가까이 있는 극점이 된다.

- 대표극점에 의한 응답이 나머지의 극점들에 의한 응답보다 적어도 3배 이상 느리면 고차 시스템의 단순 2차 시스템으로의 근사화 가능
- 유수의 크기는 시스템의 극점 및 영점의 위치에 따라 결정된다. 시스템의 극점과 영점의 위치가 근접할수록 유수는 작아지게 된다.

▶ 고차 시스템 G(s)에 대한 단위스텝응답의 일반적인 해

$$G(s) = \frac{y(s)}{u(s)} = \frac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{q}(s+p_j)\prod_{k=1}^{r}(s^2+2\zeta_k\omega_\kappa s+\omega_k^2)}$$

단위스텝입력(u(s)=1/s)을 가했을 때 출력 y(s)는

$$y(s)=\frac{a}{s}+\sum_{j=1}^{q}\frac{a_j}{s+p_j}+\sum_{k=1}^{r}\frac{b_k(s+\zeta_k\omega_k)+c_k\omega_k\sqrt{1-\zeta_k^2}}{s_2+2\zeta_k\omega_k s+\omega_k^2} \qquad (3.64)$$

위 식을 역 Laplace 변환하여 시간역에서 출력 y(t)를 구하면

$$y(t)=a+\sum_{j=1}^{q}a_j e^{-p_j t}+\sum_{k=1}^{r}b_k e^{-\zeta_k\omega_k t}\cos\omega_k\sqrt{1-\zeta_k^2}\,t$$
$$+\sum_{k=1}^{r}c_k e^{-\zeta_k\omega_k t}\sin\omega_k\sqrt{1-\zeta_k^2}\,t \qquad (t\geq 0) \qquad (3.65)$$

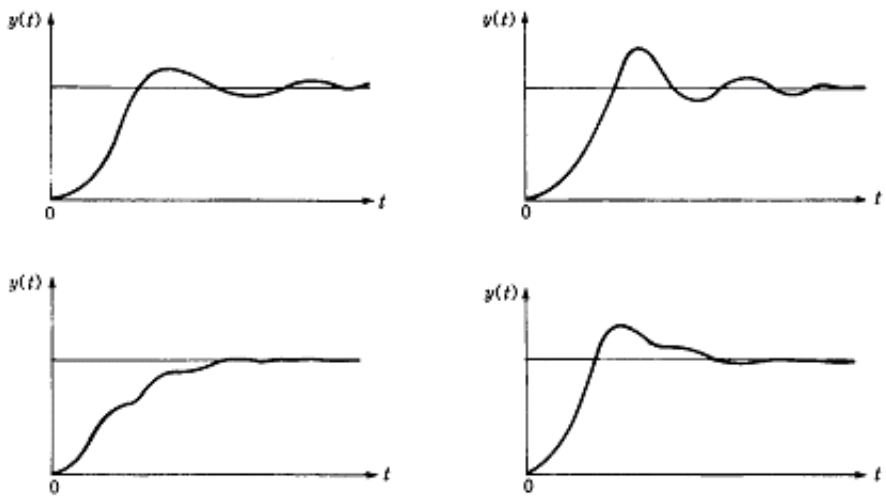


그림 3.13 고차 시스템에 대한 스텝응답의 예

$$\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d T_r + \theta)$$

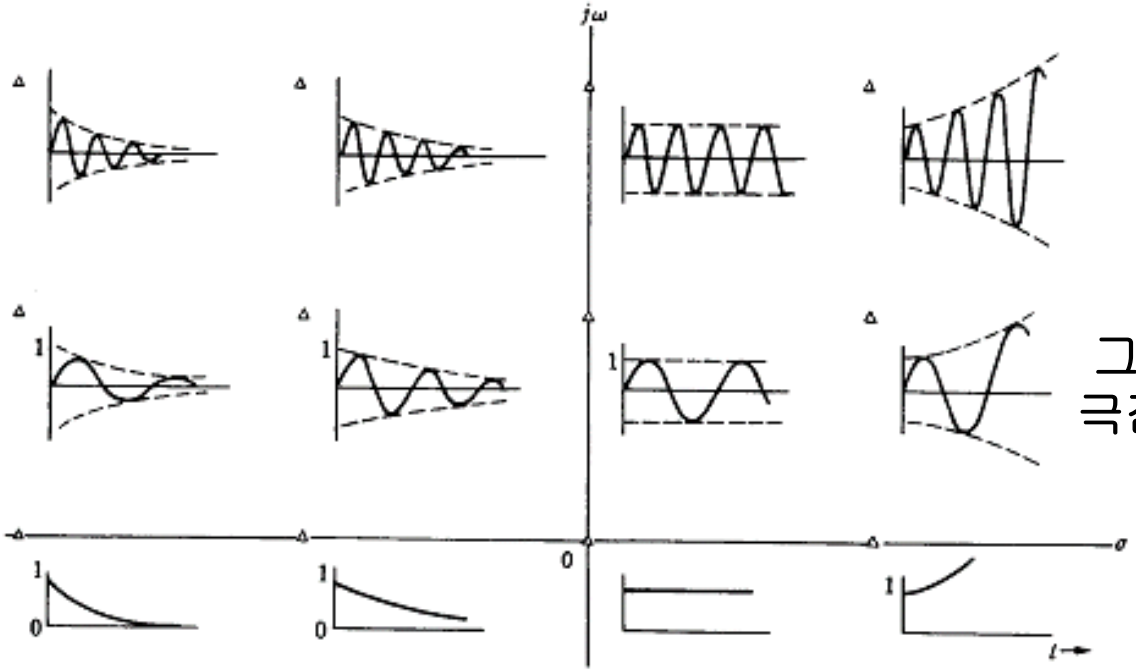


그림 3.14 s-평면상에서
극점의 위치에 따른 임펄스응답

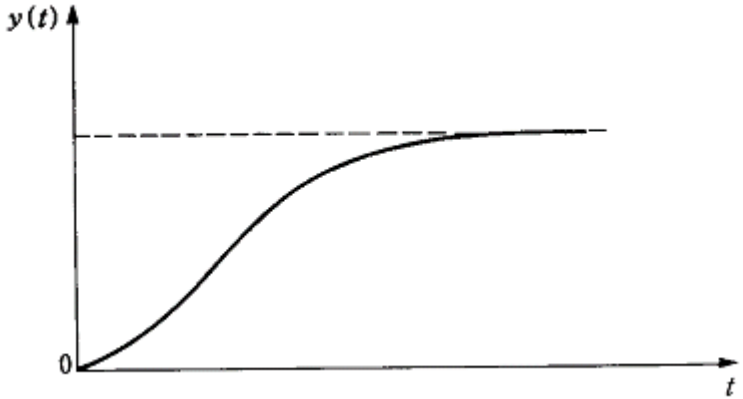


그림 3.15 스텝응답 특성

Home Work #4 (Due date: two weeks from today)

1. In a system with an input and an output, what poles generate the steady-state response?

2. If a pole is moved with a constant imaginary part, what will the responses have in common?

3. If a pole is moved with a constant real part, what will the responses have in common?

4. List five specifications for a second-order underdamped system.

5. Solve Problem 25(25) on page 211(217) in the text book 7th ed *(6th edition)

6. Solve Problem 26(26) on page 212(217) in the text book.

7. Solve Problems 31(28) on page 212(217) in the text book.

8. Solve Problem 32(29) on page 212(217) in the text book.