Chapter 3
Differential Motions and Velocities

3.1 INTRODUCTION

♦ Definition of Differential Motion : A small movements of mechanism that can be used to derive velocity relationships between different parts of the mechanism.

♦ In this chapters.......

- Differential Motions of frames relative to a fixed frame
- Jacobians and robot velocity relationships

3.2 DIFFERENTIAL RELATIONSHIPS

♦ Concept of the differential relationships :

The velocity of point *B* :

$$\bar{V}_B = \bar{V}_A + \bar{V}_{B/A} = \frac{d}{d\theta_1}\left(l_1 \cos\theta_1 \hat{i} + l_1 \sin\theta_1 \hat{j}\right)\frac{d\theta_1}{dt} +$$

$$\frac{d}{d(\theta_1+\theta_2)}\left(l_2 \cos(\theta_1+\theta_2)\hat{i} + l_2 \sin(\theta_1+\theta_2)\hat{j}\right)\frac{d(\theta_1+\theta_2)}{dt}$$

$$= -l_1\dot{\theta}_1 \sin\theta_1 \hat{i} + l_1\dot{\theta}_1 \cos\theta_1 \hat{j} - l_2(\dot{\theta}_1+\dot{\theta}_2)\sin(\theta_1+\theta_2)\hat{i} + l_2(\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_1+\theta_2)\hat{j}$$

$$\begin{bmatrix} \bar{V}_{B_x} \\ \bar{V}_{B_y} \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

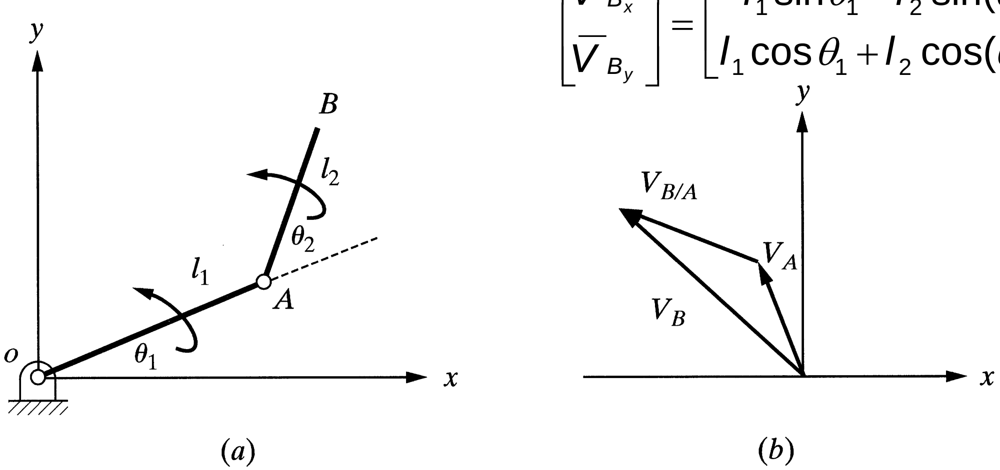

Fig. 3.1 (a) A two-degree-of-freedom planar mechanism and (b) a Velocity diagram

3.2 DIFFERENTIAL RELATIONSHIPS

Velocity relationship of point *B* :

$$\begin{cases} x_B = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ y_B = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{cases}$$

$$\begin{cases} dx_B = -l_1 \sin\theta_1 d\theta_1 - l_2 \sin(\theta_1 + \theta_2)(d\theta_1 + d\theta_2) \\ dy_B = l_1 \cos\theta_1 d\theta_1 + l_2 \cos(\theta_1 + \theta_2)(d\theta_1 + d\theta_2) \end{cases}$$

$$\begin{bmatrix} dx_B \\ dy_B \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix}$$

Differential Motion of *B* | Jacobian | Differential Motion of Joint

$$dX = Jd\Theta$$

The joint differential motions can be related to the differential motion of the hand .

3.3 JACOBIAN

- **Definition :** Jacobian is a representation of the geometry of the elements of a mechanism in time.
- **Formation :** Jacobian is formed from the elements of the position equations that were differentiated with respect to θ_1 and θ_2.
- **Assumption :** A set of equations Y_i in terms of a set of variables x_j:

$$Y_i = f_i(x_1, x_2, x_3, \cdots, x_j)$$

$$\begin{cases} \delta Y_1 = \dfrac{\partial f_1}{\partial x_1}\delta x_1 + \dfrac{\partial f_1}{\partial x_2}\delta x_2 + \cdots + \dfrac{\partial f_1}{\partial x_j}\delta x_j \\ \delta Y_2 = \dfrac{\partial f_2}{\partial x_1}\delta x_1 + \dfrac{\partial f_2}{\partial x_2}\delta x_2 + \cdots + \dfrac{\partial f_2}{\partial x_j}\delta x_j \\ \vdots \\ \delta Y_i = \dfrac{\partial f_i}{\partial x_1}\delta x_1 + \dfrac{\partial f_i}{\partial x_2}\delta x_2 + \cdots + \dfrac{\partial f_i}{\partial x_j}\delta x_j \end{cases}$$

3.3 JACOBIAN

$$\begin{bmatrix} \delta Y_1 \\ \delta Y_2 \\ \mathrm{M} \\ \delta Y_i \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \mathrm{L} & \frac{\partial f_1}{\partial x_j} \\ \frac{\partial f_2}{\partial x_1} & \mathrm{L} & \mathrm{L} & \mathrm{L} \\ \mathrm{M} & & & \\ \frac{\partial f_i}{\partial x_1} & & & \frac{\partial f_i}{\partial x_j} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \mathrm{M} \\ \delta x_j \end{bmatrix}$$

Matrix Representation

$$[\partial Y_i] = \left[\frac{\partial f_i}{\partial x_i}\right][\partial x_j]$$

$$\begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} Robot \\ Jacobian \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix}$$

Differentiating the position equations of a robot

$$[D] = [J][D_\theta]$$

3.4 DIFFERENTIAL MOTIONS OF A FRAME

- The differential motion of a hand frame of the robot are caused by the differential motions in each of the joints of the robot.

- The differential motion of a frame:
  - Differential translations,
  - Differential rotations,
  - Differential transformations(translations and rotations).

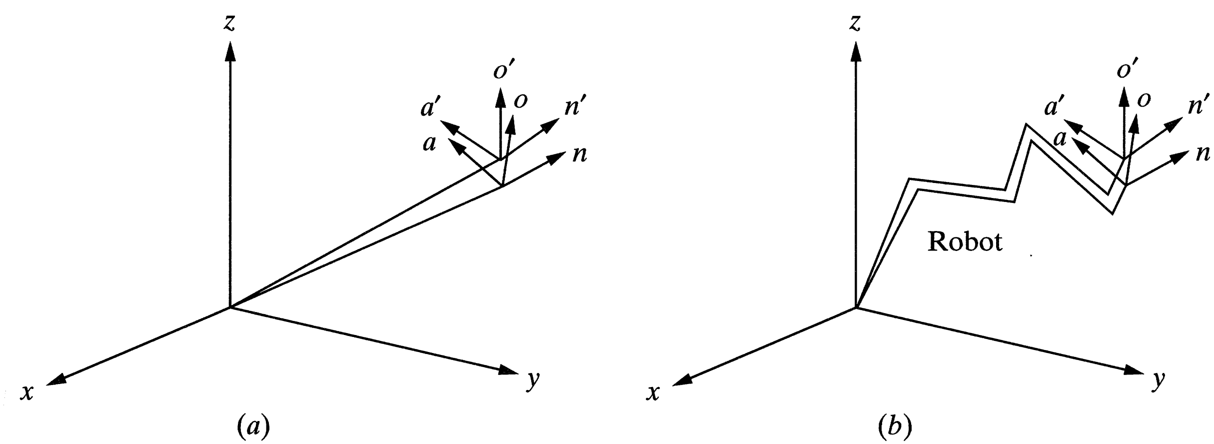

Fig. 3.2 (a) Differential motions of a frame and (b) differential motions of a frame as related to the differential motions of a robot. (b) a Velocity diagram

3.4.1 Differential Translations

- Definition : A translation of a frame at differential values.
- Representation : Trans(dx, dy, dz)

  ≈ The frame has moved a differential amount along the 3 axes.

3.4.2 Differential rotations

- Definition : A small rotation of a frame at differential values.
- Representation : Rot(k, $d\theta$) , $\sin\delta x = \delta x, \cos\delta x = 1$

≈ The frame has rotated an angle of $d\theta$ about an axis $\hat{k}$

≈ Differential rotation about the x, y, z-axis is δx, δy, δz, respectively.

$$Rot(x,\delta x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta x & 0 \\ 0 & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad Rot(y,\delta y) = \begin{bmatrix} 1 & 0 & -\delta y & 0 \\ 0 & 1 & 0 & 0 \\ \delta y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad Rot(z,\delta z) = \begin{bmatrix} 1 & -\delta z & 0 & 0 \\ \delta z & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(x,\delta x)Rot(y,\delta y) = Rot(y,\delta y)Rot(x,\delta x)$$

$$Rot(x,\delta x)Rot(y,\delta y)=\begin{bmatrix}1&0&0&0\\0&1&-\delta x&0\\0&\delta x&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&\delta y&0\\0&1&0&0\\-\delta y&0&1&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&\delta y&0\\\delta x\delta y&1&-\delta x&0\\-\delta y&\delta x&1&0\\0&0&0&1\end{bmatrix}$$

$$Rot(y,\delta y)Rot(x,\delta x)=\begin{bmatrix}1&0&\delta y&0\\0&1&0&0\\-\delta y&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&-\delta x&0\\0&\delta x&1&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&\delta x\delta y&\delta y&0\\0&1&-\delta x&0\\-\delta y&\delta x&1&0\\0&0&0&1\end{bmatrix}$$

$\sin\delta x=\delta x,\ \ \cos\delta x=1,\ \ \delta x\delta y\approx 0$ 로 둘 수 있으므로

$$Rot(x,\delta x)Rot(y,\delta y)=Rot(y,\delta y)Rot(x,\delta x)$$

3.4.3 Differential Rotation about a General Axis $\hat{k}$

♦ A differential motion about a general axis $\hat{k}$ is composed of 3 differential motions about the 3 axes, in any order.

$$Rot(k, d\theta) = Rot(x, \delta x)Rot(y, \delta y)Rot(z, \delta z)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta x & 0 \\ 0 & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\delta y & 0 \\ 0 & 1 & 0 & 0 \\ \delta y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\delta z & 0 & 0 \\ \delta z & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\delta z & \delta y & 0 \\ \delta x \delta y + \delta z & -\delta x \delta y \delta z + 1 & -\delta x & 0 \\ -\delta y + \delta x \delta z & \delta x + \delta y \delta z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\delta z & \delta y & 0 \\ \delta z & 1 & -\delta x & 0 \\ -\delta y & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Leftarrow$ Neglect the higher order differential motion ($\delta x\delta y$, $-\delta x\delta y\delta z$, $\delta x\delta z$, $\delta y\delta z$)

3.4.4 Differential Transformations of a Frame

- Definition : A combination of differential translations and rotations.

$$[T+dT]=[Trans(dx,dy,dz)\times Rot(k,d\theta)][T]$$

$$[dT]=[Trans(dx,dy,dz)\times Rot(k,d\theta)-I][T]$$

$$[dT]=[\Delta][T]$$

- Δ : Differential Operator

$$\Delta = Trans(dx,dy,dz)\times Rot(k,d\theta)-I$$

$$=\begin{bmatrix}1&0&0&dx\\0&1&0&dy\\0&0&1&dz\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&-\delta z&\delta y&0\\\delta z&1&-\delta x&0\\-\delta y&\delta x&1&0\\0&0&0&1\end{bmatrix}-\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}0&-\delta z&\delta y&dx\\\delta z&0&-\delta x&dy\\-\delta y&\delta x&0&dz\\0&0&0&0\end{bmatrix}$$

3.5 INTERPRETATION OF THE DIFFERENTIAL CHANGE

- Matrix *dT* represents the changes in a frame as a result of differential motions.
- Each element of the matrix represents the change in the corresponding element of the frame.

$$dT = \begin{bmatrix} dn_x & do_x & da_x & dp_x \\ dn_y & do_y & da_y & dp_y \\ dn_z & do_z & da_z & dp_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3.6 DIFFERENTIAL CHANGES BETWEEN FRAMES

$$[dT]=[\Delta][T]=[T][^{T}\Delta]$$

Fixed Frame   Current Frame

$$\longrightarrow \quad [T]^{-1}[\Delta][T]=[T]^{-1}[T][^{T}\Delta]$$

$$[^{T}\Delta]=[T]^{-1}[\Delta][T]$$

$$T^{-1}=\begin{bmatrix} n_x & n_y & n_z & -p\cdot n \\ o_x & o_y & o_z & -p\cdot o \\ a_x & a_y & a_z & -p\cdot a \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Delta=\begin{bmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[T^{-1}][\Delta][T]={}^{T}\Delta=\begin{bmatrix} 0 & -{}^{T}\delta z & {}^{T}\delta y & {}^{T}dx \\ {}^{T}\delta z & 0 & -{}^{T}\delta x & {}^{T}dy \\ -{}^{T}\delta y & {}^{T}\delta x & 0 & {}^{T}dz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} {}^{T}\delta_x=\bar{\delta}\cdot\bar{n} \\ {}^{T}\delta_y=\bar{\delta}\cdot\bar{o} \\ {}^{T}\delta_z=\bar{\delta}\cdot\bar{a} \\ {}^{T}d_x=\bar{n}\cdot[(\bar{\delta}\cdot\bar{p})+\bar{d}] \\ {}^{T}d_y=\bar{o}\cdot[(\bar{\delta}\cdot\bar{p})+\bar{d}] \\ {}^{T}d_z=\bar{a}\cdot[(\bar{\delta}\cdot\bar{p})+\bar{d}] \end{cases}$$

3.7 DIFFERENTIAL MOTIONS OF A ROBOT AND ITS HAND FRAME

- Relation between the differential motions of the joint of the robot and the differential motions of the hand frame and *dT*.
- It is a function of the robot's configuration and design and its instantaneous location and orientation.

$$\begin{bmatrix} dx \\ dy \\ dz \\ \partial x \\ \partial y \\ \partial z \end{bmatrix} = \begin{bmatrix} Robot \\ Jacobian \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix} \quad \text{OR} \quad [D] = [J][D_\theta]$$

3.8 CALCULATION OF THE JACOBIAN

♦ Key point : Each element in the jacobian is the derivative of a corresponding kinematic equation with respect to one of the variables.

Consult Example 2.19 and below….

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} C_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\ S_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\ S_{234}a_4 + S_{23}a_3 + S_2a_2 \\ 1 \end{bmatrix}$$

The last column of the forward kinematic equation of the robot

Taking the derivative of p_x

$$\begin{cases} \dfrac{\partial p_x}{\partial \theta_1} = J_{11} = -S_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\ \dfrac{\partial p_x}{\partial \theta_2} = J_{12} = C_1(-S_{234}a_4 - S_{23}a_3 - S_2a_2) \\ \dfrac{\partial p_x}{\partial \theta_3} = J_{13} = C_1(-S_{234}a_4 - S_{23}a_3) \\ \dfrac{\partial p_x}{\partial \theta_4} = J_{14} = C_1(-S_{234}a_4) \\ \dfrac{\partial p_x}{\partial \theta_5} = J_{15} = 0 \\ \dfrac{\partial p_x}{\partial \theta_6} = J_{16} = 0 \end{cases}$$

The first row of the Jacobian

3.8 CALCULATION OF THE JACOBIAN

- The velocity equation relative to the last frame

$$[{}^{T_6}D] = [{}^{T_6}J][D_\theta]$$

- The differential motion relationship of Equation

$$\begin{bmatrix} {}^{T_6}d_x \\ {}^{T_6}d_y \\ {}^{T_6}d_z \\ {}^{T_6}\delta_x \\ {}^{T_6}\delta_y \\ {}^{T_6}\delta_z \end{bmatrix} = \begin{bmatrix} {}^{T_6}J_{11} & {}^{T_6}J_{12} & \cdots & {}^{T_6}J_{16} \\ {}^{T_6}J_{21} & {}^{T_6}J_{22} & \cdots & {}^{T_6}J_{26} \\ {}^{T_6}J_{31} & . & \cdots & {}^{T_6}J_{36} \\ {}^{T_6}J_{41} & . & \cdots & {}^{T_6}J_{46} \\ {}^{T_6}J_{51} & . & \cdots & {}^{T_6}J_{56} \\ {}^{T_6}J_{61} & . & \cdots & {}^{T_6}J_{66} \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix}$$

R.P Paul은 i 번째 관절이 회전관절일 경우 다음의 관계식을 유도

$${}^{T_6}J_{1i} = \left(-n_x p_y + n_y p_x\right),\quad {}^{T_6}J_{2i} = \left(-o_x p_y + o_y p_x\right),\quad {}^{T_6}J_{3i} = \left(-a_x p_y + a_y p_x\right)$$

$${}^{T_6}J_{4i} = n_z,\quad {}^{T_6}J_{5i} = o_z,\quad {}^{T_6}J_{6i} = a_z$$

3.9 HOW TO RELATE THE JACOBIAN AND THE DIFFERENTIAL OPERATOR

- The differential motions of the robot's joints are ultimately related to the hand frame of the robot.

Method 1

- Equ. 3.10 and Jacobian calculate [D] matrix
- [D] contains differential motions of the hand, dx, dy, dz, δx, δy, δz.
- Equ. 3.15 used to calculate dT

Method 2

- Equ. 3.24 and Jacobian calculate $[{}^{T_6}D]$ matrix
- [D] contains differential motions of the hand, ${}^{T_6}dx$, ${}^{T_6}dy$, ${}^{T_6}dz$, ${}^{T_6}\delta x$, ${}^{T_6}\delta y$, ${}^{T_6}\delta z$.
- Equ. 3.19 used to calculate dT

3.10 INVERSE JACOBIAN

- Inverse Jacobian used to calculate the differential motions needed at the joints of the robot for a desired hand differential motion.
- Inverse Jacobian calculates how fast each joint must move so that the robot's hand will yield a desired differential motion or velocity.
- To make sure the robot follows a desired path, the joint velocities must be calculated continuously in order to ensure that the robot's hand maintains a desired velocity.

$$[D] = [J][D_\theta]$$

$$[J^{-1}][D] = [J^{-1}][J][D_\theta] \rightarrow [D_\theta] = [J^{-1}][D]$$

$$[{}^{T_6}J^{-1}][{}^{T_6}D] = [{}^{T_6}J^{-1}][{}^{T_6}J][D_\theta] \rightarrow D_\theta = [{}^{T_6}J^{-1}][{}^{T_6}D]$$