Chapter 4
Dynamic Analysis and Forces

4.1 INTRODUCTION

In this chapters…….

- The dynamics, related with accelerations, loads, masses and inertias.

$$\sum \overline{F} = m \cdot \overline{a} \qquad \sum \overline{T} = I \cdot \overline{\alpha}$$

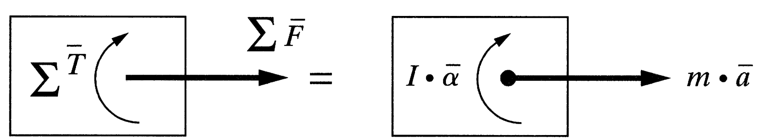


Fig. 4.1 Force-mass-acceleration and torque-inertia-angular acceleration relationships for a rigid body.

In Actuators…….

- The actuator can accelerate a robot's links for exerting enough forces and torques at a desired acceleration and velocity.
- By the dynamic relationships that govern the motions of the robot, considering the external loads, the designer can calculate the necessary forces and torques.

4.2 LAGRANGIAN MECHANICS: A SHORT OVERVIEW

- Lagrangian mechanics is based on the differentiation energy terms only, with respect to the system's variables and time.
- Definition: L = Lagrangian, K = Kinetic Energy of the system, P = Potential Energy, F = the summation of all external forces for a linear motion, T = the summation of all torques in a rotational motion, x = System variables

$$L = K - P$$

$$F_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i}$$

$$T_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}$$

$$\Rightarrow \quad \tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$$

Example 4.1 Derive the force-acceleration relationship for the one-degree of freedom system.

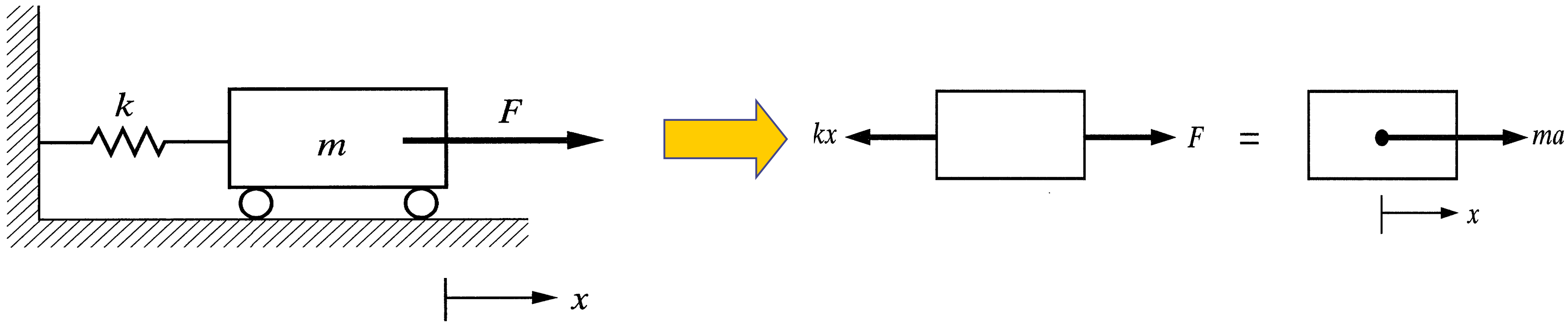


Fig. 4.2 Schematic of a simple cart-spring system.

Fig. 4.3 Free-body diagram for the sprint-cart system.

Solution

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2, \quad P = \frac{1}{2}kx^2 \quad \Rightarrow \quad L = K - P = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

- Lagrangian mechanics

$$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}, \quad \frac{d}{dt}(m\dot{x}) = m\ddot{x}, \quad \frac{\partial L}{\partial x} = -kx$$

$$F = m\ddot{x} + kx$$

- Newtonian mechanics

$$\sum \overline{F} = m \cdot \overline{a}$$

$$F - kx = ma \rightarrow F = ma + kx$$

- The complexity of the terms increases as the number of degrees of freedom and variables.

Example 4.2

Derive the equations of motion for the two-degree of freedom system.

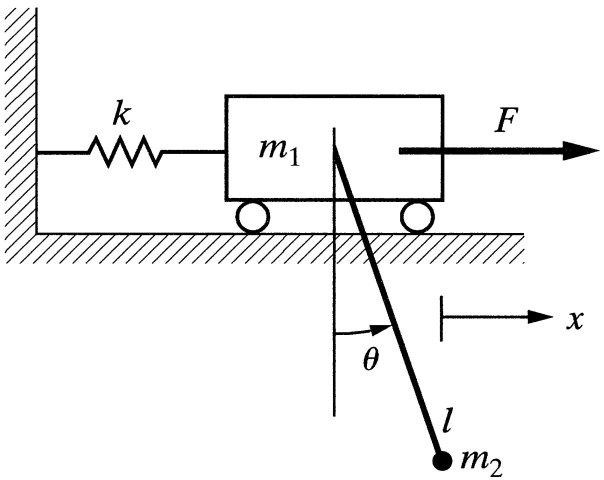


Fig. 4.4 Schematic of a cart-pendulum system.

In this system…….

- **It requires two coordinates, x and θ.**
- **It requires two equations of motion:**
  1. **The linear motion of the system.**
  2. **The rotation of the pendulum.**

Solution

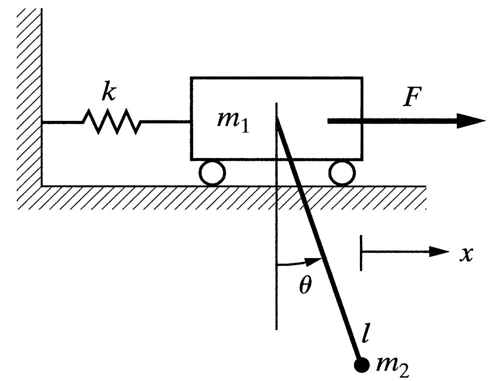
$$\begin{bmatrix} F \\ T \end{bmatrix} = \begin{bmatrix} m_1 + m_2 & m_2 l \cos\theta \\ m_2 l \cos\theta & m_2 l^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & m_2 l \sin\theta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}^2 \\ \dot{\theta}^2 \end{bmatrix} + \begin{bmatrix} kx \\ m_2 g l \sin\theta \end{bmatrix}$$

$$\bar{r}_{P/C} = l\sin\theta\,\bar{i} - l\cos\theta\,\bar{j}, \quad \bar{V}_{P/C} = \frac{d\bar{r}_{P/C}}{d\theta}\frac{d\theta}{dt}$$

The velocity of pendulum

$$\bar{V}_P = \bar{V}_C + \bar{V}_{P/C} = \dot{x}\bar{i} + l\dot{\theta}\cos\theta\,\bar{i} + l\dot{\theta}\sin\theta\,\bar{j} = \left(\dot{x} + l\dot{\theta}\cos\theta\right)\bar{i} + l\dot{\theta}\sin\theta\,\bar{j}$$

$$\bar{V}_P^{\,2} = \left(\dot{x} + l\dot{\theta}\cos\theta\right)^2 + \left(l\dot{\theta}\sin\theta\right)^2$$



The kinetic energy of the cart and pendulum

$$K = K_{cart} + K_{pendulum}$$

$$K_{cart} = \frac{1}{2}m_1\dot{x}^2$$

$$K_{pendulum} = \frac{1}{2}m_2\bar{V}_P^{\,2} = \frac{1}{2}m_2\left(\dot{x} + l\dot{\theta}\cos\theta\right)^2 + \frac{1}{2}m_2\left(l\dot{\theta}\sin\theta\right)^2$$

$$\therefore K = \frac{1}{2}\left(m_1 + m_2\right)\dot{x}^2 + \frac{1}{2}m_2\left(l^2\dot{\theta}^2 + 2l\dot{\theta}\dot{x}\cos\theta\right)$$

- The potential energy of the spring and the pendulum

$$P = \frac{1}{2}kx^2 + m_2 gl\left(1 - \cos\theta\right)$$

- The Lagrangian

$$L = K - P = \frac{1}{2}\left(m_1 + m_2\right)\dot{x}^2 + \frac{1}{2}m_2\left(l^2\dot{\theta}^2 + 2l\dot{\theta}\dot{x}\cos\theta\right) - \frac{1}{2}kx^2 - m_2 gl\left(1 - \cos\theta\right)$$

$$\frac{\partial L}{\partial \dot{x}} = \left(m_1 + m_2\right)\dot{x} + m_2 l\dot{\theta}\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \left(m_1 + m_2\right)\ddot{x} + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta$$

$$\frac{\partial L}{\partial x} = -kx$$

$$\therefore F = \left(m_1 + m_2\right)\ddot{x} + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta + kx$$

The Lagrangian for the rotational motion

$$L = K - P = \frac{1}{2}\left(m_1 + m_2\right)\dot{x}^2 + \frac{1}{2}m_2\left(l^2\dot{\theta}^2 + 2l\dot{\theta}\dot{x}\cos\theta\right) - \frac{1}{2}kx^2 - m_2 gl\left(1-\cos\theta\right)$$

$$\frac{\partial L}{\partial\dot{\theta}} = m_2 l^2\dot{\theta} + m_2 l\dot{x}\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = m_2 l^2\ddot{\theta} + m_2 l\ddot{x}\cos\theta - m_2 l\dot{x}\dot{\theta}\sin\theta$$

$$\frac{\partial L}{\partial\theta} = -m_2 l\dot{x}\dot{\theta}\sin\theta - m_2 gl\sin\theta$$

$$F = \left(m_1 + m_2\right)\ddot{x} + m_2 l\ddot{\theta}\cos\theta - m_2 l\dot{\theta}^2\sin\theta + kx$$

$$\therefore T = m_2 l^2\ddot{\theta} + m_2 l\ddot{x}\cos\theta \cancel{- m_2 l\dot{x}\dot{\theta}\sin\theta} + m_2 gl\sin\theta \cancel{+ m_2 l\dot{x}\dot{\theta}\sin\theta}$$

$$= m_2 l^2\ddot{\theta} + m_2 l\ddot{x}\cos\theta + m_2 gl\sin\theta$$

$$\therefore \begin{bmatrix} F \\ T \end{bmatrix} = \begin{bmatrix} m_1 + m_2 & m_2 l\cos\theta \\ m_2 l\cos\theta & m_2 l^2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & -m_2 l\sin\theta \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}^2 \\ \dot{\theta}^2 \end{bmatrix} + \begin{bmatrix} kx \\ m_2 gl\sin\theta \end{bmatrix}$$

Example 4.4 Using the Lagrangian method, derive the equations of motion for the two-degree of freedom robot arm.

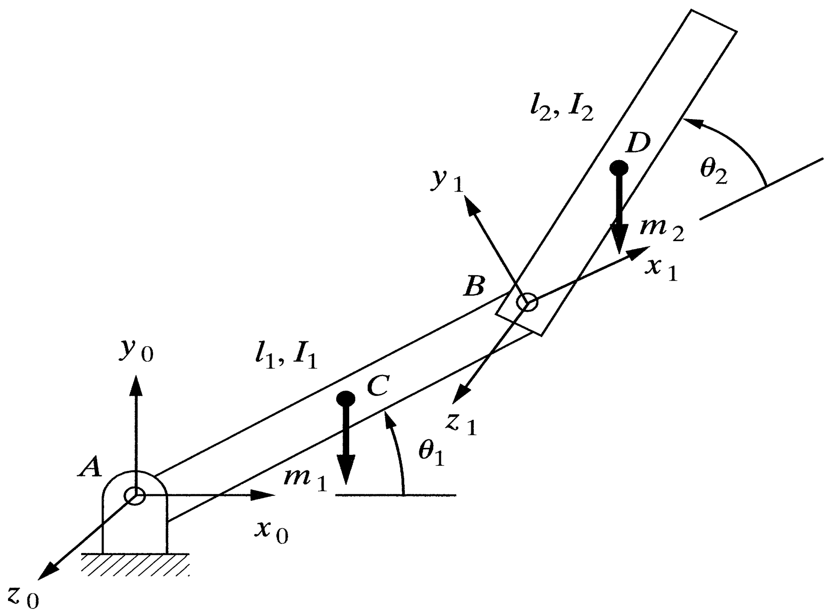


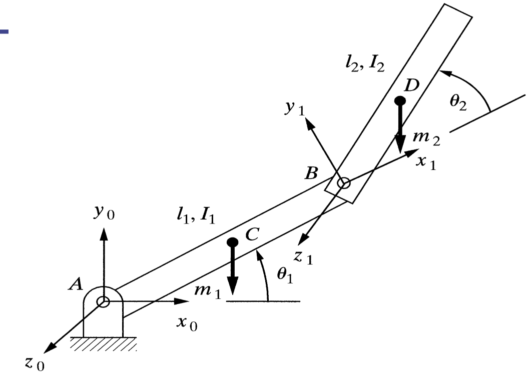
Fig. 4.6 A two-degree-of-freedom robot arm.

Solution

Follow the same steps as before…….

- **Calculates the velocity of the center of mass of link 2 by differentiating its position:**
- **The kinetic energy of the total system is the sum of the kinetic energies of links 1 and 2.**
- **The potential energy of the system is the sum of the potential energies of the two links:**

$$K = K_1 + K_2 = \left[\frac{1}{2} I_A \dot{\theta}_1^2\right] + \left[\frac{1}{2} I_D (\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2} m_2 V_D^2\right]$$



The kinetic energy of link1 and link2

$$x_D = l_1 C_1 + 0.5 l_2 C_{12} \quad \rightarrow \dot{x}_D = -l_1 S_1 \dot{\theta}_1 - 0.5 l_2 S_{12} \left(\dot{\theta}_1 + \dot{\theta}_2 \right)$$

$$y_D = l_1 S_1 + 0.5 l_2 S_{12} \quad \rightarrow \dot{y}_D = l_1 C_1 \dot{\theta}_1 + 0.5 l_2 C_{12} \left(\dot{\theta}_1 + \dot{\theta}_2 \right)$$

$$\left(\begin{aligned} V_D^2 &= \dot{x}_D^2 + \dot{y}_D^2 \\ &= \dot{\theta}_1^2 (l_1^2 + 0.25 l_2^2 + l_1 l_2 C_2) + \dot{\theta}_2^2 (0.25 l_2^2) + \dot{\theta}_1 \dot{\theta}_2 (0.5 l_2^2 + l_1 l_2 C_2) \end{aligned} \right)$$

$$\begin{aligned} K = K_1 + K_2 &= \left[\frac{1}{2} I_A \dot{\theta}_1^2 \right] + \left[\frac{1}{2} I_D (\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2} m_2 V_D^2 \right] \\ &= \left[\frac{1}{2} \left(\frac{1}{3} m_1 l_1^2 \right) \dot{\theta}_1^2 \right] + \left[\frac{1}{2} \left(\frac{1}{12} m_2 l_2^2 \right) (\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2} m_2 V_D^2 \right] \end{aligned}$$

$$I_A = \int_0^{l_1} r^2 dm = \int_0^{l_1} r^2 \rho A dr = \frac{1}{3} m_1 l_1^2, \quad I_D = \int_{-l_2/2}^{l_2/2} r^2 dm = \int_{-l_2/2}^{l_2/2} r^2 \rho A dr = \frac{1}{12} m_2 l_2^2$$

$$K = \dot{\theta}_1^2\left(\frac{1}{6}m_1 l_1^2 + \frac{1}{6}m_2 l_2^2 + \frac{1}{2}m_2 l_1^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right) + \dot{\theta}_2^2\left(\frac{1}{6}m_2 l_2^2\right) + \dot{\theta}_1\dot{\theta}_2\left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right)$$

- The potential energy

$$P = m_1 g\frac{l_1}{2}S_1 + m_2 g\left(l_1 S_1 + \frac{l_2}{2}S_{12}\right)$$

- The Lagrangian

$$L = K - P = \dot{\theta}_1^2\left(\frac{1}{6}m_1 l_1^2 + \frac{1}{6}m_2 l_2^2 + \frac{1}{2}m_2 l_1^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right) + \dot{\theta}_2^2\left(\frac{1}{6}m_2 l_2^2\right) + \dot{\theta}_1\dot{\theta}_2\left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right) - m_1 g\frac{l_1}{2}S_1 - m_2 g\left(l_1 S_1 + \frac{l_2}{2}S_{12}\right)$$

$$L = K - P = \dot{\theta}_1^2\left(\frac{1}{6}m_1 l_1^2 + \frac{1}{6}m_2 l_2^2 + \frac{1}{2}m_2 l_1^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right) + \dot{\theta}_2^2\left(\frac{1}{6}m_2 l_2^2\right)$$
$$+ \dot{\theta}_1\dot{\theta}_2\left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right) - m_1 g\frac{l_1}{2}S_1 - m_2 g\left(l_1 S_1 + \frac{l_2}{2}S_{12}\right)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = 2\left(\frac{1}{6}m_1 l_1^2 + \frac{1}{6}m_2 l_2^2 + \frac{1}{2}m_2 l_1^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right)\dot{\theta}_1 + \left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right)\dot{\theta}_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \left(\frac{1}{3}m_1 l_1^2 + \frac{1}{3}m_2 l_2^2 + m_2 l_1^2 + m_2 l_1 l_2 C_2\right)\ddot{\theta}_1 - m_2 l_1 l_2 S_2\,\dot{\theta}_1\dot{\theta}_2$$
$$+ \left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right)\ddot{\theta}_2 - \frac{1}{2}m_2 l_1 l_2 S_2\dot{\theta}_2^2$$

$$\frac{\partial L}{\partial \theta_1} = -\left(\frac{1}{2}m_1 + m_2\right)g l_1 C_1 - \frac{1}{2}m_2 g l_2 C_{12}$$

$$T_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}$$

From $T_1 = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}_1}\right) - \dfrac{\partial L}{\partial \theta_1}$

$$T_1 = \left(\frac{1}{3}m_1 l_1^2 + m_2 l_1^2 + \frac{1}{3}m_2 l_2^2 + m_2 l_1 l_2 C_2\right)\ddot{\theta}_1 + \left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right)\ddot{\theta}_2$$

$$-(m_2 l_1 l_2 S_2)\dot{\theta}_1\dot{\theta}_2 - \left(\frac{1}{2}m_2 l_1 l_2 S_2\right)\dot{\theta}_2^2 + \left(\frac{1}{2}m_1 + m_2\right) g l_1 C_1 + \frac{1}{2}m_2 g l_2 C_{12}$$

From $T_2 = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}_2}\right) - \dfrac{\partial L}{\partial \theta_2}$

$$T_2 = \left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right)\ddot{\theta}_1 + \left(\frac{1}{3}m_2 l_2^2\right)\ddot{\theta}_2 + \left(\frac{1}{2}m_2 l_1 l_2 S_2\right)\dot{\theta}_1^2 + \frac{1}{2}m_2 g l_2 C_{12}$$

4.3 EFFECTIVE MOMENTS OF INERTIA

- To Simplify the equation of motion, Equations can be rewritten in symbolic form.

$$T_1 = \left(\frac{1}{3}m_1 l_1^2 + m_2 l_1^2 + \frac{1}{3}m_2 l_2^2 + m_2 l_1 l_2 C_2\right)\ddot{\theta}_1 + \left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right)\ddot{\theta}_2$$

$$-(m_2 l_1 l_2 S_2)\dot{\theta}_1\dot{\theta}_2 - \left(\frac{1}{2}m_2 l_1 l_2 S_2\right)\dot{\theta}_2^2 + \left(\frac{1}{2}m_1 + m_2\right)g l_1 C_1 + \frac{1}{2}m_2 g l_2 C_{12}$$

$$T_2 = \left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right)\ddot{\theta}_1 + \left(\frac{1}{3}m_2 l_2^2\right)\ddot{\theta}_2 + \left(\frac{1}{2}m_2 l_1 l_2 S_2\right)\dot{\theta}_1^2 + \frac{1}{2}m_2 g l_2 C_{12}$$

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} D_{ii} & D_{ij} \\ D_{ji} & D_{jj} \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} D_{iii} & D_{ijj} \\ D_{jii} & D_{jjj} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} D_{iii} & D_{ijj} \\ D_{jii} & D_{jjj} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_2\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

4.4 DYNAMIC EQUATIONS FOR MULTIPLE-DEGREE-OF-FREEDOM ROBOTS

4.4.1 Kinetic Energy

- Equations for a multiple-degree-of-freedom robot are very long and complicated, but can be found by calculating the kinetic and potential energies of the links and the joints, by defining the Lagrangian and by differentiating the Lagrangian equation with respect to the joint variables.

  - **The kinetic energy of a rigid body with motion in three dimension :**

$$K = \frac{1}{2} m \overline{V}^2 + \frac{1}{2} \overline{\omega}\, \overline{h}_G$$

  - **The kinetic energy of a rigid body in planar motion**

$$K = \frac{1}{2} m \overline{V}^2 + \frac{1}{2} \overline{I} \omega^2$$

$\overline{h}_G$ is the angular momentum of the body about G

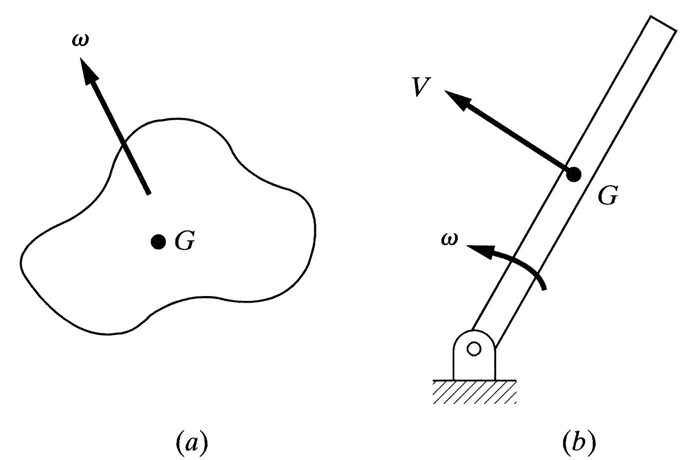


Fig. 4.7 A rigid body in three-dimensional motion and in plane motion.

4.4 DYNAMIC EQUATIONS FOR MULTIPLE-DEGREE-OF-FREEDOM ROBOTS

4.4.1 Kinetic Energy

- **The velocity of a point along a robot's link can be defined by differentiating the position equation of the point.**

$$p_i = {}^{R}T_i \, {}^{i}r_i = {}^{0}T_i \, {}^{i}r_i$$

$$V_i = \frac{d}{dt}\left({}^{R}T_i \, {}^{i}r_i\right) = \sum_{j=1}^{i}\left(\frac{\partial\left({}^{0}T_i\right)}{\partial q_j}\frac{dq_j}{dt}\right)\cdot {}^{i}r_i$$

- **The kinetic energy along a robot's link can be derived by**

$$K_i = \frac{1}{2}\sum_{i=1}^{n}\sum_{p=1}^{i}\sum_{r=1}^{i} Trace\left(U_{ip} J_i U_{ir}^{T}\right)\dot{q}_p \dot{q}_r + \frac{1}{2}\sum_{i=1}^{n} I_{i(act)}\dot{q}_i^{2}$$

4.4 DYNAMIC EQUATIONS FOR MULTIPLE-DEGREE-OF-FREEDOM ROBOTS

4.4.2 Potential Energy

- **The potential energy of the system is the sum of the potential energies of each link.**

$$P = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} [-m_i g^T \cdot ({}^0T_i \bar{r}_i)]$$

$$where\ g^T = \begin{bmatrix} g_x & g_y & g_z & 0 \end{bmatrix}^T$$

- **The potential energy must be a scalar quantity and the values in the gravity matrix are dependent on the orientation of the reference frame.**

4.4 DYNAMIC EQUATIONS FOR MULTIPLE-DEGREE-OF-FREEDOM ROBOTS

4.4.3 The Lagrangian

$$L = K - P = \frac{1}{2}\sum_{i=1}^{n}\sum_{p=1}^{i}\sum_{r=1}^{i} Trace\left(U_{ip} J_i U_{ir}^{T}\right)\dot{q}_p \dot{q}_r$$

$$+\frac{1}{2}\sum_{i=1}^{n} I_{i(act)}\dot{q}_i^{2} - \sum_{i=1}^{n}\left[-m_i g^{T}\cdot\left({}^{0}T_i \bar{r}_i\right)\right]$$

4.4 DYNAMIC EQUATIONS FOR MULTIPLE-DEGREE-OF-FREEDOM ROBOTS

4.4.4 Robot's Equations of Motion

- **The Lagrangian is differentiated to form the dynamic equations of motion.**
- **The final equations of motion for a general multi-axis robot is below.**

$$T_i = \sum_{j=1}^{n} D_{ij}\ddot{q}_j + I_{i(act)}\ddot{q}_i + \sum_{j=1}^{n}\sum_{k=1}^{n} D_{ijk}\dot{q}_j\dot{q}_k + D_i$$

where, $D_{ij} = \sum_{p=\max(i,j)}^{n} Trace(U_{pj} J_p U_{pi}^T)$

$$D_{ijk} = \sum_{p=\max(i,j,k)}^{n} Trace(U_{pjk} J_p U_{pi}^T)$$

$$D_i = \sum_{p=i}^{n} -m_p g^T U_{pi} \bar{r}_p$$

Joint velocity of a Robot manipulator

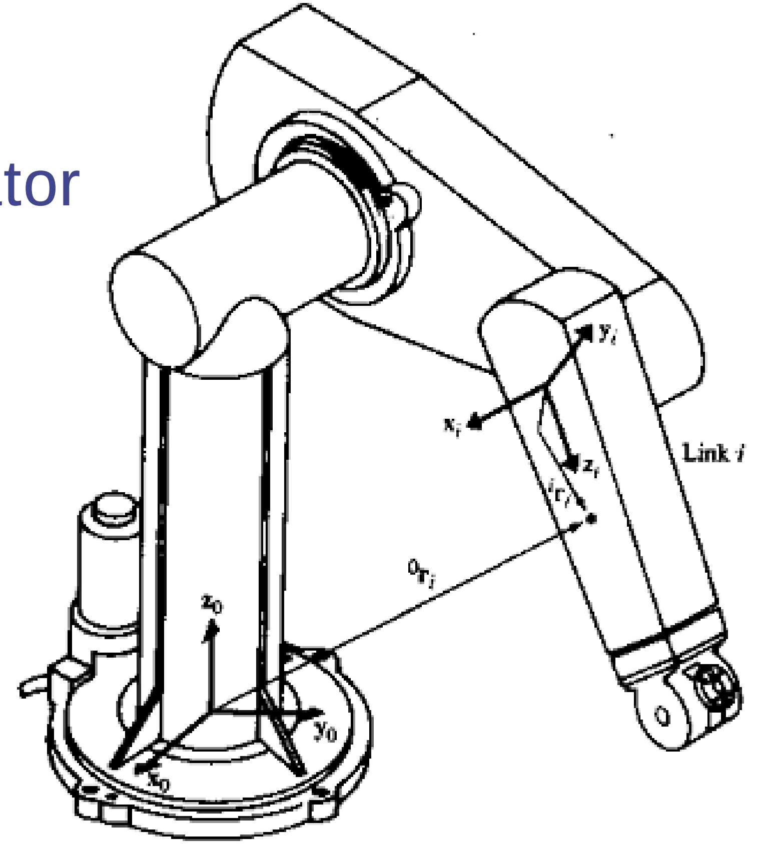


Figure 3.1 A point $^i\mathbf{r}_i$ in link i.

$$^{i-1}r_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = [x_i,\ y_i,\ z_i,\ 1]^T$$

$$^0r_i = {}^0A_i\,{}^ir_i$$

where, $^0A_i = {}^0A_1\,{}^1A_2 \cdots {}^{i\text{-}1}A_i$

$$^{i-1}A_i = T_{z,d}T_{z,\theta}T_{x,a}T_{x,\alpha}$$

$$= \begin{bmatrix} \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\alpha_i\cos\theta_i & -\sin\alpha_i\cos\theta_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
{}^0v_i \equiv v_i = \frac{d}{dt}({}^0r_i) = \frac{d}{dt}({}^0A_i\,{}^ir_i)
$$

$$
= {}^0\dot{A}_1\,{}^1A_2 \cdots {}^{i-1}A_i\,{}^ir_i + {}^0A_1\,{}^1\dot{A}_2 \cdots {}^{i-1}A_i\,{}^ir_i + \cdots
$$

$$
+ {}^0A_1 \cdots {}^{i-1}\dot{A}_i\,{}^ir_i + {}^0A_i\,{}^i\dot{r}_i
$$

$$
= \left[\sum_{j=1}^{i} \frac{\partial\,{}^0A_i}{\partial q_j}\,\dot{q}_j\right]{}^ir_i
$$

The partial differential of ${}^0A_i$ related to q_i is easily derived by using matrix Q_i

* rotation joint

$$
Q_i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

* prismatic joint

$$
Q_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$ {}^{i-1}A_i = T_{z,d}T_{z,\theta}T_{x,a}T_{x,\alpha} = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\alpha_i\cos\theta_i & -\sin\alpha_i\cos\theta_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ \frac{\partial\, {}^{i-1}A_i}{\partial q_i} = Q_i\, {}^{i-1}A_i $$

$$ \frac{\partial\, {}^{i-1}A_i}{\partial \theta_i} = \begin{bmatrix} -\sin\theta_i & -\cos\alpha_i\cos\theta_i & \sin\alpha_i\cos\theta_i & -a_i\sin\theta_i \\ \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i & a_i\cos\theta_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

$$ = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\alpha_i\cos\theta_i & -\sin\alpha_i\cos\theta_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv Q_i\, {}^{i-1}A_i $$

$$ \therefore \frac{\partial\, {}^0A_i}{\partial q_j} = \begin{cases} {}^0A_1\, {}^1A_2 \cdots {}^{\text{j-2}}A_{j-1} Q_j\, {}^{j-1}A_j \cdots {}^{i-1}A_i & j \le i \\ 0 & j > i \end{cases} $$

where we define $\mathrm{U}_{\mathrm{ij}} \equiv \dfrac{\partial\, {}^0A_i}{\partial q_j}$

$$\mathrm{U}_{\mathrm{ij}} = \begin{cases} {}^{0}A_{j-1} Q_j \, {}^{j-1}A_i & j \le i \\ 0 & j > i \end{cases}$$

$$v_i = \left[\sum_{j=1}^{i} \frac{\partial\, {}^{0}A_i}{\partial q_j} \dot{q}_j \right] {}^{i}r_i = \left[\sum_{j=1}^{i} U_{ij} \dot{q}_j \right] {}^{i}r_i$$

where velocity is also derived by using matrix ${}^{i-1}A_i$ and Q_i

Next, we find the interaction effects between joints as

$$\frac{\partial U_{ij}}{\partial q_k} \cong U_{ijk} = \begin{cases} {}^{0}A_{j-1} Q_j \, {}^{j-1}A_{k-1} Q_k \, {}^{k-1}A_i & i \ge k \ge j \\ {}^{0}A_{k-1} Q_k \, {}^{k-1}A_{j-1} Q_j \, {}^{j-1}A_i & i \ge j \ge k \\ 0 & i < j \;\; or \;\; i < k \end{cases}$$

Kinetic Energy of a Robot Manipulator

- The kinetic energy dK_i of a particle with differential mass dm

$$dK_i = \frac{1}{2}(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)dm$$

$$v_i = \left[\sum_{j=1}^{i} U_{ij}\dot{q}_j\right] {}^i r_i$$

$$= \frac{1}{2} trace(v_i v_i^T)dm = \frac{1}{2} Tr(v_i v_i^T)dm$$

$$\text{where } \left(\mathrm{Tr}\, A \equiv \sum_{i=1}^{n} a_{ii}\right),\ {}^i r_i = (x_i,\ y_i,\ z_i,\ 1)^T$$

$$({}^0\dot{r}_i)^2 = {}^0\dot{r}_i^T \cdot {}^0\dot{r}_i = trace({}^0\dot{r}_i \cdot {}^0\dot{r}_i^T)$$

$$dK_i = \frac{1}{2} Tr\left[\sum_{j=1}^{i} U_{ij}\dot{q}_j {}^i r_i \left[\sum_{k=1}^{i} U_{ik}\dot{q}_k {}^i r_i\right]^T\right]dm$$

$$= \frac{1}{2} Tr\left[\sum_{j=1}^{i}\sum_{k=1}^{i} U_{ij} {}^i r_i {}^i r_i^T U_{ik}^T \dot{q}_j \dot{q}_k\right]dm$$

$$= \frac{1}{2} Tr\left[\sum_{j=1}^{i}\sum_{k=1}^{i} U_{ij} ({}^i r_i {}^i r_i^T dm) U_{ik}^T \dot{q}_j \dot{q}_k\right]$$

U_{ir} and $\dot{q}_i$ are independent of the mass distribution of *link i*.

The kinetic energies of all links

$$K_i = \int dK_i = \frac{1}{2} Tr[\sum_{j=1}^{i}\sum_{k=1}^{i} U_{ij} (\int {}^i r_i \, {}^i r_i^T dm) U_{ik}^T \dot{q}_j \dot{q}_k]$$

$$\therefore J_i = \int {}^i r_i \, {}^i r_i^T dm = \begin{bmatrix} \int x_i^2 dm & \int x_i y_i dm & \int x_i z_i dm & \int x_i dm \\ \int x_i y_i dm & \int y_i^2 dm & \int y_i z_i dm & \int y_i dm \\ \int x_i z_i dm & \int y_i z_i dm & \int z_i^2 dm & \int z_i dm \\ \int x_i dm & \int y_i dm & \int z_i dm & \int dm \end{bmatrix}$$

where, ${}^i r_i = (x_i, \; y_i, \; z_i, \; 1)^T$.

$$K = \sum_{i=1}^{n} K_i = \frac{1}{2}\sum_{i=1}^{n} Tr[\sum_{j=1}^{i}\sum_{k=1}^{i} U_{ij} J_i U_{ik}^T \dot{q}_j \dot{q}_k]$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} [Tr(U_{ij} J_i U_{ik}^T) \dot{q}_j \dot{q}_k]$$

Potential Energy of a Robot Manipulator

$$P_i = -m_i g({}^{0}\bar{r}_i) = -m_i g({}^{0}A_i\,{}^{i}\bar{r}_i) \quad , \quad i = 1,\ 2,\ \ldots\ ,\ n$$

$$\therefore P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} (-m_i) g({}^{0}A_i\,{}^{i}\bar{r}_i)$$

where , $g = (g_x,\ g_y,\ g_z,\ 0)$

in horizontal system $g = (0,\ 0,\ -|g|,\ 0)$

Motion Equations of a Manipulator

$$L = K - P$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i}[Tr(U_{ij}J_iU_{ik}{}^T)\dot{q}_j\dot{q}_k] + \sum_{i=1}^{n} m_i g({}^0A_i\,{}^i\bar{r}_i)$$

$$\tau_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$$

$$= \sum_{j=i}^{n}\sum_{k=1}^{j} Tr(U_{jk}J_jU_{ji}{}^T)\ddot{q}_k + \sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} Tr(U_{jkm}J_jU_{ji}{}^T)\dot{q}_k\dot{q}_m$$

$$-\sum_{j=i}^{n} m_j g U_{ji}\,{}^j\bar{r}_j$$

$$\tau_i = \sum_{k=1}^{n} D_{ik}\ddot{q}_k + \sum_{k=1}^{n}\sum_{m=1}^{n} h_{ikm}\dot{q}_k\dot{q}_m + c_i \qquad, i = 1, 2, \ldots\ldots, n$$

※ Appendix for derivation of motion equation of manipulator

$$L=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i}[Tr(U_{ij}J_iU_{ik}{}^T)\dot{q}_j\dot{q}_k]+\sum_{i=1}^{n}m_i g({}^0A_i\,{}^i\bar{r}_i)$$

$$\frac{\partial L}{\partial \dot{q}_i}=\frac{1}{2}\left\{\sum_{j=i}^{n}\sum_{k=1}^{j}[Tr(U_{ji}J_jU_{jk}{}^T)]\frac{\partial \dot{q}_i}{\partial \dot{q}_i}\dot{q}_k\right.$$

$$\left.+\sum_{k=i}^{n}\sum_{j=1}^{k}[Tr(U_{kj}J_kU_{ki}{}^T)]\frac{\partial \dot{q}_i}{\partial \dot{q}_i}\dot{q}_j\right\}$$

In second term of right side, if the subcripts are substituted as

$j->k,\ k->j$ and $Tr(AB^T)=Tr(AB^T)^T=Tr(BA^T),\quad J_k=J_k{}^T$ are used

$$\sum_{k=i}^{n}\sum_{j=1}^{k}[Tr(U_{kj}J_kU_{ki}{}^T)]=\sum_{j=i}^{n}\sum_{k=1}^{j}[Tr(U_{jk}J_jU_{ji}{}^T)]=\sum_{j=i}^{n}\sum_{k=1}^{j}[Tr(U_{ji}J_jU_{jk}{}^T)]$$

$$\therefore\frac{\partial L}{\partial \dot{q}_i}=\sum_{j=i}^{n}\sum_{k=1}^{j}[Tr(U_{ji}J_jU_{jk}{}^T)]\dot{q}_k=\sum_{k=i}^{n}\sum_{j=1}^{k}[Tr(U_{kj}J_kU_{ki}{}^T)]\dot{q}_j$$

$$\frac{\partial L}{\partial \dot{q}_i} = \sum_{j=i}^{n}\sum_{k=1}^{j}[Tr(U_{ji}J_jU_{jk}{}^T)]\,\dot{q}_k = \sum_{j=i}^{n}\sum_{k=1}^{j}Tr(U_{jk}J_jU_{ji}{}^T)\dot{q}_k$$

Since k is increased to j, equation $\dfrac{\partial ※}{\partial q_m}$ is 0 when m is over j

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \sum_{m=1}^{j}\frac{\partial}{\partial q_m}\left[\sum_{j=i}^{n}\sum_{k=1}^{j}Tr(U_{jk}J_jU_{ji}{}^T)\dot{q}_k\right]\frac{dq_m}{dt}$$

$$+\sum_{j=i}^{n}\sum_{k=1}^{j}Tr(U_{jk}J_jU_{ji}{}^T)\ddot{q}_k$$

$$=\sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j}[Tr(U_{jkm}J_jU_{ji}{}^T)\dot{q}_k\dot{q}_m + Tr(U_{jk}J_jU_{jim}{}^T)\dot{q}_k\dot{q}_m]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{j}Tr(U_{jk}J_jU_{ji}{}^T)\ddot{q}_k$$

$$=2\sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j}[Tr(U_{jkm}J_jU_{ji}{}^T)\dot{q}_k\dot{q}_m]+\sum_{j=1}^{n}\sum_{k=1}^{j}Tr(U_{jk}J_jU_{ji}{}^T)\ddot{q}_k$$

- As same way, subscript is substituted as $i->j,\ \ j->k,\ \ k->m$

$$L=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i}[Tr(U_{ij}J_{i}U_{ik}^{T})\dot{q}_j\dot{q}_k]+\sum_{i=1}^{n}m_i g({}^{0}A_i\,{}^{i}\bar{r}_i)$$

$$\frac{\partial L}{\partial q_i}=\frac{1}{2}\sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j}\frac{\partial}{\partial q_i}[Tr(U_{jk}J_{j}U_{jm}^{T})\dot{q}_k\dot{q}_m]+\sum_{j=i}^{n}m_j g\frac{\partial\,{}^{0}A_j}{\partial q_i}{}^{j}\bar{r}_j$$

$$=\frac{1}{2}\sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j}[Tr(U_{jki}J_{j}U_{jm}^{T}+U_{jk}J_{j}U_{jmi}^{T})]\dot{q}_k\dot{q}_m+\sum_{j=i}^{n}m_j g U_{ji}\,{}^{j}\bar{r}_j$$

$$=\sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j}[Tr(U_{jki}J_{j}U_{jm}^{T})\dot{q}_k\dot{q}_m]+\sum_{j=i}^{n}m_j g U_{ji}\,{}^{j}\bar{r}_j$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)=2\sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j}[Tr(U_{jkm}J_{j}U_{ji}^{T})\dot{q}_k\dot{q}_m+\sum_{j=1}^{n}\sum_{k=1}^{j}Tr(U_{jk}J_{j}U_{ji}^{T})\ddot{q}_k$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i$$ 에서

$$※ \sum_{j=i}^{n}\sum_{k=1}^{j} Tr(U_{jk} J_j U_{ji}{}^T)\ddot{q}_k + \sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} [Tr(U_{jkm} J_j U_{ji}{}^T)\dot{q}_k \dot{q}_m$$

$$- \sum_{j=i}^{n} m_j g U_{ji}{}^{j}\bar{r}_j = \tau_i$$

$$\therefore \sum_{k=1}^{n} D_{ik}\ddot{q}_k + \sum_{k=1}^{n}\sum_{m=1}^{n} h_{ikm}\dot{q}_k \dot{q}_m + c_i = \tau_i$$

단, $D_{ik} = \sum_{j=\max(i,k)}^{n} Tr(U_{jk} J_j U_{ji}{}^T)$

$$h_{ikm} = \sum_{j=\max(i,k,m)}^{n} Tr(U_{jkm} J_j U_{ji}{}^T)$$

If a robot has 6 DOF, the dynamic model is derived as

$$\tau_i(t) = \sum_{k=i}^{6}\sum_{j=1}^{k} Tr\left[\frac{\partial\, {}^0T_k}{\partial q_j} J_k \left(\frac{\partial\, {}^0T_k}{\partial q_i}\right)^T\right]\ddot{q}_j(t)$$

$$+\sum_{r=i}^{6}\sum_{j=1}^{r}\sum_{k=1}^{r} Tr\left[\frac{\partial^2\, {}^0T_r}{\partial q_j \partial q_k} J_r \left(\frac{\partial\, {}^0T_r}{\partial q_i}\right)^T\right]\dot{q}_j\dot{q}_k - \sum_{j=i}^{6} m_j g\left(\frac{\partial\, {}^0T_j}{\partial q_i}\right)\bar{r}_j$$

- General Robot Dynamic Model

$$\tau(t) = D(q(t))\ddot{q}(t) + h(q(t),\ \dot{q}(t)) + C(q(t))$$

$\tau(t)$: $n\times 1$ *generalized torque vector*

$q(t)$: $n\times 1$ *vector of the joint variables of the robot arm*

$\dot{q}(t)$: $n\times 1$ *vector of the joint velocity of the robot arm*

$\ddot{q}(t)$: $n\times 1$ *vector of the joint acceleration of the robot arm*

$$D(q):\ D_{ik} = \sum_{j=\max(i,k)}^{n} Tr(U_{jk} J_j U_{ji}^{T}) \ :\ inertia\ acceleration,$$

$moment$ 에 관계되는 term

$h(q, \dot{q})$: $Coriolis\ \ force,\ centrifugal\ \ force$ 에 관련되는 term

$$h(q, \dot{q}) = (h_1, h_2, \cdots, h_n)^T$$

$$h_i = \sum_{k=1}^{n} \sum_{m=1}^{n} h_{ikm} \dot{q}_k \dot{q}_m$$

$$h_{ikm} = \sum_{j=\max(i,k,m)}^{n} Tr(U_{jkm} J_j U_{ji}^{T})$$

$C(q)$: $gravity\ loading\ \ force\ vector$ $(n \times 1)$

$$C_i = \sum_{j=i}^{n} (-m_j g U_{ji}\, {}^{j}\bar{r}_j)$$

★ Exmaple: 2 Link manipulator

$$\tau_i = \sum_{k=1}^{n} D_{ik}\ddot{q}_k + \sum_{k=1}^{n}\sum_{m=1}^{n} h_{ikm}\dot{q}_k\dot{q}_m + C_i$$

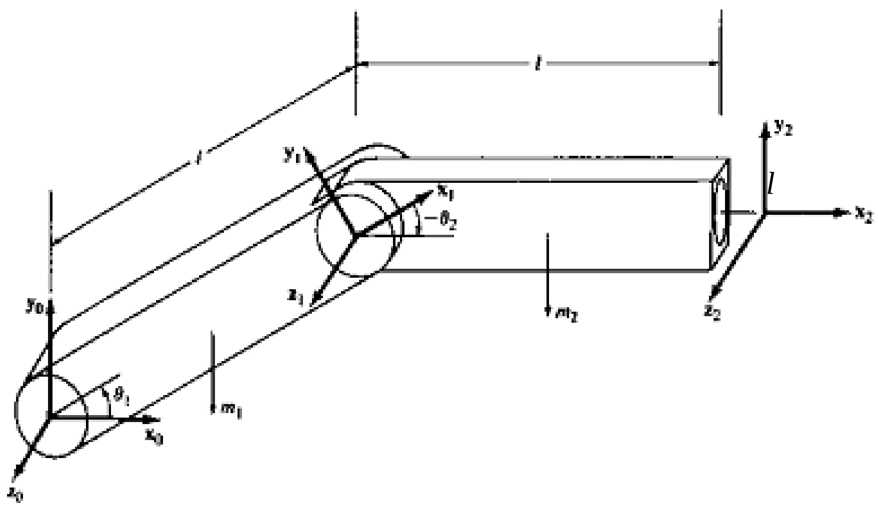


Figure 3.2 A two-link manipulator.

link	θ	d	a	α
1	θ_1	0	l	0
2	θ_2	0	l	0

$$\alpha_1 = \alpha_2 = 0, \quad d_1 = d_2 = 0, \quad a_1 = a_2 = l$$

link	θ	d	a	α
1	θ_1	0	l	0
2	θ_2	0	l	0

$$^{i-1}A_i = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0A_1 = T_{z,d} T_{z,\theta} T_{z,a} T_{x,\alpha}$$

$$= \begin{bmatrix} C_1 & -S_1 & 0 & lC_1 \\ S_1 & C_1 & 0 & lS_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^1A_2 = \begin{bmatrix} C_2 & -S_2 & 0 & lC_2 \\ S_2 & C_2 & 0 & lS_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore\ ^0A_2 = {}^0A_1\,{}^1A_2 = \begin{bmatrix} C_{12} & -S_{12} & 0 & l(C_{12}+C_1) \\ S_{12} & C_{12} & 0 & l(S_{12}+S_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the definition of the Q_i matrix for a rotary joint i

For deriving $D_{ik} = \sum_{j=\max(i,k)}^{n} Tr(U_{jk} J_j U_{ji}^T)$

$$ {}^0A_1 = \begin{bmatrix} C_1 & -S_1 & 0 & lC_1 \\ S_1 & C_1 & 0 & lS_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ U_{11} = \frac{\partial\, {}^0A_1}{\partial \theta_1} = Q_1\, {}^0A_1 = \begin{bmatrix} -S_1 & -C_1 & 0 & -lS_1 \\ C_1 & -S_1 & 0 & lS_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

$$ U_{21} = \frac{\partial\, {}^0A_2}{\partial \theta_1} = Q_1\, {}^0A_2 = \begin{bmatrix} -S_{12} & -C_{12} & 0 & -l(S_{12}+S_1) \\ C_{12} & -S_{12} & 0 & l(C_{12}+C_1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

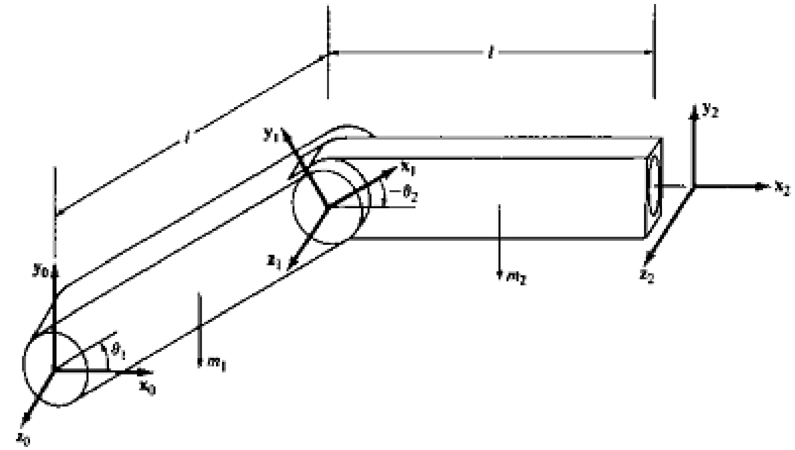
$$ U_{22} = \frac{\partial\, {}^0A_2}{\partial \theta_2} = {}^0A_1 Q_2\, {}^1A_2 $$

$$ = \begin{bmatrix} C_1 & -S_1 & 0 & lC_1 \\ S_1 & C_1 & 0 & lS_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -S_2 & -C_2 & 0 & -lS_2 \\ C_2 & -S_2 & 0 & lC_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -S_{12} & -C_{12} & 0 & -lS_{12} \\ C_{12} & -S_{12} & 0 & lC_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

Eq (3.2-18)에서 $I_{xy} = I_{xz} = I_{yz} = 0$

$$I_{1x} = \int x_1^2 dm$$

$$= \int_0^{-l} x^2 \rho A(-dx) = \rho A \frac{l^3}{3} = \frac{1}{3} m_1 l^2$$

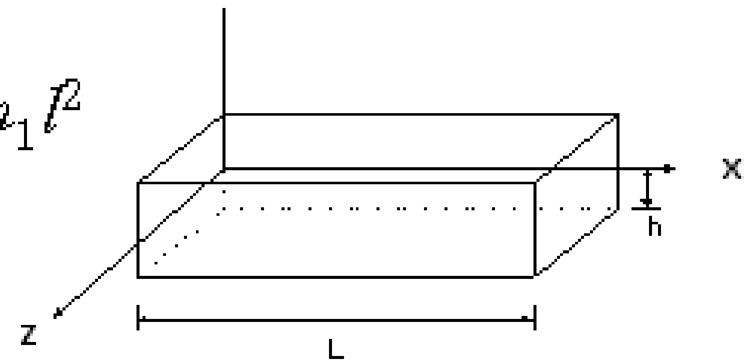


$$I_{2x} = \frac{1}{3} m_2 l^2$$

$$I_y = I_z = 0$$

(Q)

$m_1 l^2$



$$m = \rho A l$$

$$I_x = \int_0^l x^2 \rho A dx = \rho A \frac{l^3}{3} = \frac{1}{3} m l^2$$

$$I_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} y_i^2 dm = \int_{-\frac{h}{2}}^{\frac{h}{2}} y_i^2 \rho A dh = \frac{2}{3} \frac{h^3}{8} \rho A = \frac{1}{12} m h^2 \approx 0$$

$$J_1 = \begin{bmatrix} \frac{1}{3}m_1 l^2 & 0 & 0 & -\frac{1}{2}m_1 l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}m_1 l & 0 & 0 & m_1 \end{bmatrix} \qquad J_2 = \begin{bmatrix} \frac{1}{3}m_2 l^2 & 0 & 0 & -\frac{1}{2}m_2 l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}m_2 l & 0 & 0 & m_2 \end{bmatrix}$$

$$D_{ik} = \sum_{j=\max(i,k)}^{n} Tr(U_{jk} J_j U_{ji}^T)$$

$$D_{11} = Tr(U_{11}J_1U_{11}^T) + Tr(U_{21}J_2U_{21}^T)$$

$$= \frac{1}{3}m_1 l^2 + \frac{4}{3}m_2 l^2 + m_2 C_2 l^2$$

$$D_{12} = D_{21} = Tr(U_{22}J_2U_{21}^T)]$$

$$= m_2 l^2 (-\frac{1}{6} + \frac{1}{2} + \frac{1}{2}C_2) = \frac{1}{3}m_2 l^2 + \frac{1}{2}m_2 l^2 C_2$$

$$D_{22} = Tr(U_{22}J_2U_{22}^T) = \frac{1}{3}m_2 l^2 S_{12}^2 + \frac{1}{3}m_2 l^2 C_{12}^2 = \frac{1}{3}m_2 l^2$$

▶ Colioris and Centrifugal terms

$$U_{11} = \begin{bmatrix} -S_1 & -C_1 & 0 & -lS_1 \\ C_1 & -S_1 & 0 & lS_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$h_{ikm} = \sum_{j=\max(i,\,k,\,m)}^{n} Tr(U_{jkm} J_j U_{ji}{}^T)$$ 에서

$$h_1 = \sum_{k=1}^{2}\sum_{m=1}^{2} h_{1km}\dot{\theta}_k\dot{\theta}_m = h_{111}\dot{\theta}_1^2 + h_{112}\dot{\theta}_1\dot{\theta}_2 + h_{121}\dot{\theta}_1\dot{\theta}_2 + h_{122}\dot{\theta}_2^2$$

$$h_{111} = \sum_{j=1}^{2} Tr(U_{j11} J_j U_{j1}{}^T) = Tr(U_{111} J_1 U_{11}{}^T) + Tr(U_{211} J_2 U_{21}{}^T)$$

$$h_{112} = \sum_{j=2}^{2} Tr(U_{j12} J_j U_{j1}{}^T) = Tr(U_{212} J_2 U_{21}{}^T)$$

$$h_{121} = Tr(U_{221} J_2 U_{21}{}^T)$$

$$h_{122} = Tr(U_{222} J_2 U_{21}{}^T)$$

$$\therefore h_1 = -\frac{1}{2} m_2 S_2 l^2 \dot{\theta}_2^2 - m_2 S_2 l^2 \dot{\theta}_1\dot{\theta}_2$$

$$h_2 = \sum_{k=1}^{2}\sum_{m=1}^{2} h_{2km}\dot{\theta}_k\dot{\theta}_m = h_{211}\dot{\theta}_1^2 + h_{212}\dot{\theta}_1\dot{\theta}_2 + h_{221}\dot{\theta}_1 + h_{222}\dot{\theta}_2^2$$

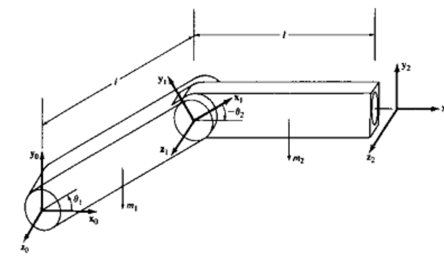
$$= \frac{1}{2} m_2 S_2 l^2 \dot{\theta}_1^2$$

$$\therefore h(\theta, \dot{\theta}) = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} m_2 S_2 l^2 \dot{\theta}_2^2 - m_2 S_2 l^2 \dot{\theta}_1\dot{\theta}_2 \\ \frac{1}{2} m_2 S_2 l^2 \dot{\theta}_1^2 \end{bmatrix}$$

- gravity-related terms

$$c = (c_1,\ c_2)^T$$

$$c_i = \sum_{j=i}^{n} (-m_j g U_{ji}\, {}^{j}\bar{r}_j) \quad \rightarrow \quad \sum_{j=i}^{2} (-m_j g U_{ji}\, {}^{j}\bar{r}_j)$$



$$c_1 = -(m_1 g U_{11}\,{}^1\bar{r}_1 + m_2 g U_{21}\,{}^2\bar{r}_2)$$

$$= -m_1(0,\,-g,\,0,\,0)\begin{bmatrix} -S_1 & -C_1 & 0 & -lS_1 \\ C_1 & -S_1 & 0 & lS_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{l}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$-m_2(0,\,-g,\,0,\,0)\begin{bmatrix} -S_{12} & -C_{12} & 0 & -l(S_{12}+S_1) \\ C_{12} & -S_{12} & 0 & l(C_{12}+C_1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{l}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}m_1 g l C_1 + \frac{1}{2}m_2 g l C_{12} + \frac{1}{2}m_2 g l C_1$$

$$c_2 = -m_2 g U_{22} {}^2\bar{r}_2$$

$$= -m_2 (0,\ -g,\ 0,\ 0) \begin{bmatrix} -S_{12} & -C_{12} & 0 & -lS_{12} \\ C_{12} & -S_{12} & 0 & lC_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{l}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= -m_2 \left(\frac{1}{2} glC_{12} - glC_{12}\right)$$

$$\therefore c(\theta) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} m_1 glC_1 + \frac{1}{2} m_2 glC_{12} + \frac{1}{2} m_2 glC_1 \\ \frac{1}{2} m_2 glC_{12} \end{bmatrix}$$

$$\therefore \tau(t) = D(\theta)\ddot{\theta}(t) + h(\theta,\, \dot{\theta}) + c(\theta) \text{ 에서}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}m_1 l^2 + \frac{4}{3}m_2 l^2 + m_2 C_2 l^2 & \frac{1}{3}m_2 l^2 + \frac{1}{2}m_2 l^2 C_2 \\ \frac{1}{3}m_2 l^2 + \frac{1}{2}m_2 l^2 C_2 & \frac{1}{3}m_2 l^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -\frac{1}{2}m_2 S_2 l^2 \dot{\theta}_2^2 - m_2 S_2 l^2 \dot{\theta}_1 \dot{\theta}_2 \\ \frac{1}{2}m_2 S_2 l^2 \dot{\theta}_1^2 \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{1}{2}m_1 g l C_1 + \frac{1}{2}m_2 g l C_{12} + \frac{1}{2}m_2 g l C_1 \\ \frac{1}{2}m_2 g l C_{12} \end{bmatrix}$$

Example 4.7 Using the aforementioned equations, derive the equations of motion for the two-degree of freedom robot arm. The two links are assumed to be of equal length.

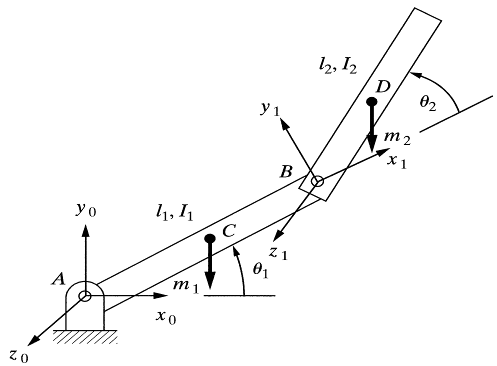


Fig. 4.8 The two-degree-of-freedom robot arm of Example 4.4

Follow the same steps as before…….

- **Write the A matrices for the two links;**
- **Develop the D_{ij} , D_{ijk} and D_i for the robot.**

Solution

- **The final equations of motion without the actuator inertia terms are the same as below.**

$$T_1=\left(\frac{1}{3}m_1l^2+\frac{4}{3}m_2l^2+m_2l^2C_2\right)\ddot{\theta}_1+\left(\frac{1}{3}m_2l^2+\frac{1}{2}m_2l^2C_2\right)\ddot{\theta}_2$$

$$+\left(\frac{1}{2}m_2l^2S_2\right)\dot{\theta}_2^2+\left(m_2l^2S_2\right)\dot{\theta}_1\dot{\theta}_2+\frac{1}{2}m_1glC_1+\frac{1}{2}m_2glC_{12}+m_2glC_1+I_{1(act)}\ddot{\theta}_1$$

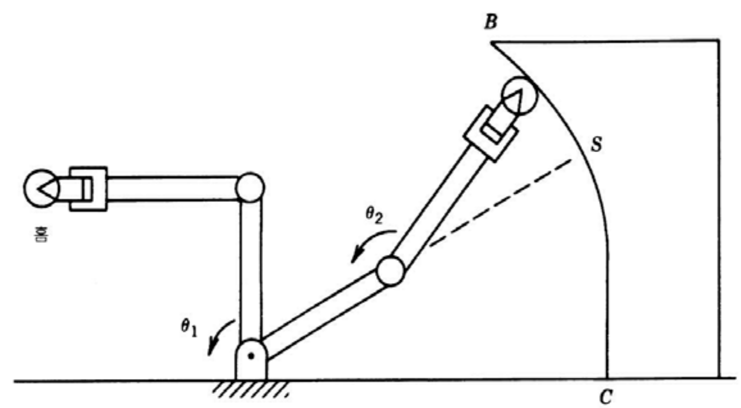
$$T_2=\left(\frac{1}{3}m_2l^2+\frac{1}{2}m_2l^2C_2\right)\ddot{\theta}_1+\left(\frac{1}{3}m_2l^2\right)\ddot{\theta}_2+\left(\frac{1}{2}m_2l^2S_2\right)+\frac{1}{2}m_2glC_{12}+I_{2(act)}\ddot{\theta}_1$$

4.5 STATIC FORCE ANALYSIS OF ROBOTS

- **Robot Control means Position Control and Force Control.**
- **Position Control:** **The robot follows a prescribed path without any reactive force.**
- **Force Control:** **The robot encounters with unknown surfaces and manages to handle the task by adjusting the uniform depth while getting the reactive force.**

Ex) Tapping a Hole - move the joints and rotate them at particular rates to create the desired forces and moments at the hand frame.

Ex) Peg Insertion – avoid the jamming while guiding the peg into the hole and inserting it to the desired depth.



4.5 STATIC FORCE ANALYSIS OF ROBOTS

- **To relate the joint forces and torques to forces and moments generated at the hand frame of the robot.**

$$\left[{}^{H}F\right]=\left[f_x \quad f_y \quad f_z \quad m_x \quad m_y \quad m_z\right]^T$$

- **F is the force and m is the moment along the axes of the hand frame.**

$$\delta W=\left[{}^{H}F\right]^T\left[{}^{H}D\right]=\left[T\right]^T\left[D_\theta\right]$$

- **The total virtual work at the joints must be the same as the total work at the hand frame.**

$$\left[{}^{H}F\right]^T\left[{}^{H}D\right]=\left[f_x \quad f_y \quad f_z \quad m_x \quad m_y \quad m_z\right]\begin{bmatrix} dx \\ dy \\ dz \\ \partial x \\ \partial y \\ \partial z \end{bmatrix}=f_x dx+\mathrm{L}\ \ \mathrm{L}\ +m_z\delta z$$

From Eq. (3.10) and (3. 24)

$$\left[{}^{T_6}D\right]=\left[{}^{T_6}J\right]\left[D_\theta\right] \quad \text{또는} \quad \left[{}^{H}D\right]=\left[{}^{H}J\right]\left[D_\theta\right]$$

From Eq. (4. 56): $\delta W=\left[{}^{H}F\right]^T\left[{}^{H}D\right]=\left[T\right]^T\left[D_\theta\right]$

$$\left[{}^{H}F\right]^T\left[{}^{H}D\right]=\left[{}^{H}F\right]^T\left[{}^{H}J\right]\left[D_\theta\right]=\left[T\right]^T\left[D_\theta\right]$$

$$\therefore \left[{}^{H}F\right]^T\left[{}^{H}J\right]=\left[T\right]^T$$

$$\therefore \left[T\right]=\left[{}^{H}J\right]^T\left[{}^{H}F\right]$$

$$\left[T\right]=\left[{}^{H}J\right]^T\left[{}^{H}F\right]$$

- **Referring to Appendix A**

4.6 TRANSFORMATION OF FORCES AND MOMENTS BETWEEN COORDINATE FRAMES

- **An equivalent force and moment with respect to the other coordinate frame by the principle of virtual work.**

$$[F]^T = [f_x \;\; f_y \;\; f_z \;\; m_x \;\; m_y \;\; m_z] \qquad \Rightarrow \qquad [{}^B F]^T = [{}^B f_x \;\; {}^B f_y \;\; {}^B f_z \;\; {}^B m_x \;\; {}^B m_y \;\; {}^B m_z]$$

$$[D]^T = [d_x \;\; d_y \;\; d_z \;\; \delta_x \;\; \delta_y \;\; \delta_z] \qquad\qquad [{}^B D]^T = [{}^B d_x \;\; {}^B d_y \;\; {}^B d_z \;\; {}^B \delta_x \;\; {}^B \delta_y \;\; {}^B \delta_z]$$

- **The total virtual work performed on the object in either frame must be the same.**

$$\delta W = [F]^T [D] = [{}^B T]^T [{}^B D]$$

4.6 TRANSFORMATION OF FORCES AND MOMENTS BETWEEN COORDINATE FRAMES

- **Displacements relative to the two frames are related to each other by the following relationship.**

$$\left[{}^{B}D\right]=\left[{}^{B}J\right][D]$$

$$\delta W=[F]^{T}[D]=\left[{}^{B}F\right]^{T}\left[{}^{B}D\right]=\left[{}^{B}F\right]^{T}\left[{}^{B}J\right][D]$$

$$\therefore [F]^{T}=\left[{}^{B}F\right]\left[{}^{B}J\right] \quad \Rightarrow [F]=\left[{}^{B}J\right]^{T}\left[{}^{B}F\right]$$

- **The forces and moments with respect to frame *B* is can be calculated directly from the following equations:**

$$^{B}f_x=\bar{n}\cdot\bar{f} \qquad {}^{B}m_x=\bar{n}\cdot[\left(\bar{f}\times\bar{p}\right)+\bar{m}]$$

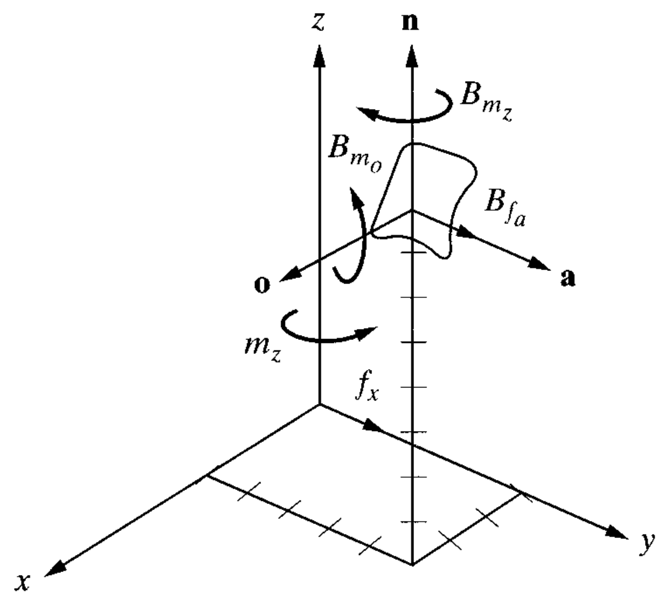
$$^{B}f_y=\bar{o}\cdot\bar{f} \qquad {}^{B}m_y=\bar{o}\cdot[\left(\bar{f}\times\bar{p}\right)+\bar{m}]$$

$$^{B}f_z=\bar{a}\cdot\bar{f} \qquad {}^{B}m_z=\bar{a}\cdot[\left(\bar{f}\times\bar{p}\right)+\bar{m}]$$

Ex 4.9 Find the equivalent forces and moments in frame B

$$F^T = \begin{bmatrix} f_x & f_y & f_z & m_x & m_y & m_z \end{bmatrix} = \begin{bmatrix} 0 & 10lb & 0 & 0 & 0 & 20lb \cdot in \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Solution

$$\overline{f} = [0 \quad 10 \quad 0], \quad \overline{m} = [0 \quad 0 \quad 20], \quad \overline{p} = [3 \quad 5 \quad 8]$$

$$\overline{n} = [0 \quad 0 \quad 1], \quad \overline{o} = [1 \quad 0 \quad 0], \quad \overline{a} = [0 \quad 1 \quad 0]$$

$$\overline{f} \times \overline{p} = \begin{vmatrix} i & j & k \\ 0 & 10 & 0 \\ 3 & 5 & 8 \end{vmatrix} = 80i - 0j - 30k$$

$$\overline{f} \times \overline{p} + \overline{m} = 80i - 10k$$

★ Appendix (Lagrange-Euler Formation)

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right]-\frac{\partial L}{\partial q_i}=\tau_i \qquad i=1,\ 2,\ \cdots,\ n$$

$L=K-P$: *Lagrange funtion*

K : *total kinetic energy*

P : *total potential energy*

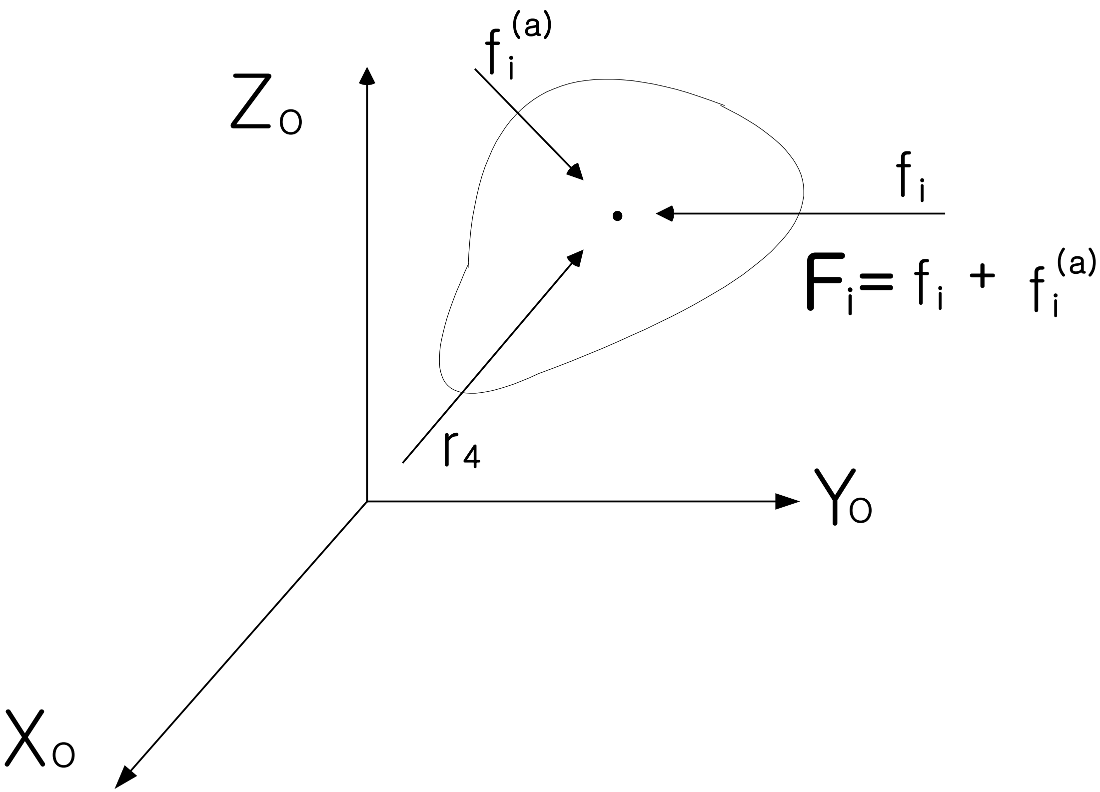
τ_i : *generalized force*

In case of rotary joint, $q_i \equiv Q_i$

In case of prismatic joint, $q_i \equiv d_i$

▶ Derivation of Lagrange-Euler Equation

- If we consider independent generalized coordinate $(q_1, q_2, \cdots, q_n)$ on a particle of mass, the position vector of particle mass m_i is expressed by $r_i = r_i(q_1, \cdots, q_n)$



If virtual displacement which is infinitesimal displacement, $\delta \mathrm{r}_\mathrm{i}$, is considered.

$$(r_1 - r_2)^T (r_1 - r_2)$$

$$\approx (r_1 + \delta r_1 - r_2 - \delta r_2)^T (r_1 - \delta r_1 - r_2 - \delta r_2) = l^2$$

$$\therefore (r_1 - r_2)^T (r_1 - r_2) + 2(r_1 - r_2)^T (\delta r_1 - \delta r_2)$$

$$+ (\delta r_1 - \delta r_2)^T (\delta r_1 - \delta r_2) = l^2$$

$$\therefore 2(r_1 - r_2)^T (\delta r_1 - \delta r_2) = 0$$

Using a generalized coordinate

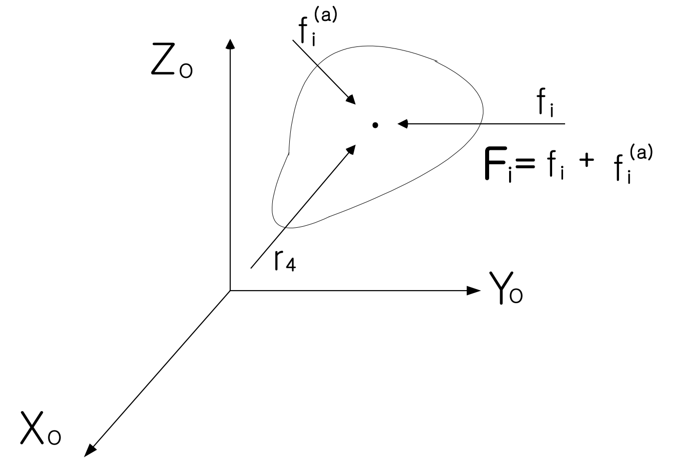
$$\delta r_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j$$

(δq_j : virtual displacement in generalized coordinate, q_j : coordinate)

The total virtual work generated by virtual displacement and force F equals zero.

That is, $\sum_{i=1}^{k} F_i^T \delta r_i = 0$ $\quad F_i$: Total force worked on particle i

$$F_i : f_i + f_i^{(a)}$$



where total work generated by $f_i^{(a)}$

$$\sum_{i=1}^{k} (f_i^{(a)})^T \delta r_i = 0 \Rightarrow \quad \sum_{i=1}^{k} (f_i)^T \delta r_i = 0$$

If we consider a virtual force $-\dot{p}_i (= m_i \ddot{r}_i)$ and D'alembert's principle, each particle satisfies an equilibrium equation such as

$$\sum (F_i - \dot{p}_i)\delta r_i = 0$$

When the virtual work is applied to a system and delete constraint force

because $\sum_{i=1}^{k} (f_i^{(a)})^T \delta r_i = 0.$

$$\left(\sum_{i=1}^{k} f_i^T - \sum_{i=1}^{k} \dot{p}_i^T\right)\delta r_i = 0 \quad \therefore \sum_{i=1}^{k} f_i^T \delta r_i - \sum_{i=1}^{k} \dot{p}_i^T \delta r_i = 0 \qquad -①$$

From ①, the virtual work generated by f_i

$$\sum_{i=1}^{k} f_i^T \delta r_i = \sum_{i=1}^{k}\sum_{j=1}^{n} f_i^T \frac{\partial r_i}{\partial q_j}\delta q_j = \sum_{j=1}^{n} \psi_j \delta q_j \qquad -②$$

where, $\delta r_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j}\delta q_j, \;\; \psi_j = \sum_{i=1}^{k} f_i^T \frac{\partial r_i}{\partial q_j}$ (j th generalized force)

A differential equation of $p_i = m_i \dot{r}_i$ in equation ① is derived as

$$\sum_{i=1}^{k} \dot{p}_i^T \delta r_i = \sum_{i=1}^{k} m_i \ddot{r}_i^T \delta r_i = \sum_{i=1}^{k} \sum_{j=1}^{n} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} \delta q_j \qquad -③$$

where

$$\sum_{i=1}^{k} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{k} \left\{ \frac{d}{dt}\left[m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j} \right] - m_i \dot{r}_i^T \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j} \right] \right\} \qquad -④$$

A velocity v_i of particle m_i

$$v_i = \dot{r}_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \frac{dq_j}{dt} = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \dot{q}_j$$

$$\therefore \frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j} \left(\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \dot{q}_j \right) = \frac{\partial r_i}{\partial q_j} \qquad -⑤$$

And

$$\frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j} \right] = \sum_{j=1}^{n} \frac{\partial^2 r_i}{\partial q_j \partial q_l} \dot{q}_l = \frac{\partial v_i}{\partial q_j} \qquad \left[\leftarrow \frac{d}{dt}\left(\frac{\partial v_i}{\partial q_j} dt \right) \right] \qquad -⑥$$

Equation ⑤, ⑥ are substituted to Eq. ④,

$$\sum_{i=1}^{k} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{k} \left\{ \frac{d}{dt}\left[m_i v_i^T \frac{\partial v_i}{\partial \dot{q}_j} \right] - m_i v_i^T \frac{\partial v_i}{\partial q_j} \right\} \qquad -⑦$$

The kinetic energy K is as

$$K = \frac{1}{2}\sum_{i=1}^{k} m_i v_i^T v_i \qquad \therefore \frac{\partial K}{\partial \dot{q}_j} = \sum_{i=1}^{k} m_i v_i^T \frac{\partial v_i}{\partial \dot{q}_j}$$

Equation ⑦ is derived as

$$\sum_{i=1}^{k} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} \qquad -⑧$$

$$\sum_{i=1}^{k} f_i^T \delta r_i - \sum_{i=1}^{k} \dot{p}_i^T \delta r_i = 0$$

Therefore from equation ③ and ⑧

$$\sum_{i=1}^{k} \dot{p}_i^T \delta r_i = \sum_{i=1}^{k} m_i \ddot{r}_i \delta r_i = \sum_{j=1}^{n}\sum_{i=1}^{k} \left(m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j}\right)\delta q_j = \sum_{j=1}^{n}\left(\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j}\right)\delta q_j \qquad -⑨$$

Using eqations ② and ⑨, equation ① is derived as

$$\therefore \sum_{i=1}^{k} \dot{p}_i^T \delta r_i - \sum_{i=1}^{k} f_i^T \delta r_i = \sum_{i=1}^{k} \dot{p}_i^T \delta r_i - \sum_{j=1}^{n} \psi_j \delta q_j = 0 \qquad -⑩$$

$$\therefore \sum_{j=1}^{n} \left(\frac{d}{dt} \frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} - \psi_j \right) \delta q_j = 0$$

$$where,\ \sum_{i=1}^{k} f_i^T \delta r_i = \sum_{i=1}^{k} \sum_{j=1}^{n} f_i^T \frac{\partial r_i}{\partial q_j} \delta q_j = \sum_{j=1}^{n} \psi_j \delta q_j \qquad -②$$

$$\psi_j = \sum_{i=1}^{k} f_i^T \frac{\partial r_i}{\partial q_j}$$ 이다.

From equation ⑩

$$\therefore \frac{d}{dt} \frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} = \psi_j \qquad -⑪$$

ψ_j with potential energy P is as

$$\psi_j = -\frac{\partial P}{\partial q_j} + \tau_j \qquad - ⑫$$

$$\therefore \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} = -\frac{\partial P}{\partial q_j} + \tau_j$$

Using Lagrangian function $L = K - P$

$$\therefore \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \tau_j \qquad \left(\because \frac{\partial P}{\partial \dot{q}_j} = 0\right)$$

- The generalized Lagrange-Euler equation is as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i$$

H.W. Assignment(#2)

- Issued : April 8th, 2019
- Due : April 22nd, 2019
- Text Problems : Text (Niku) Problem 3.