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ON SLIDING OBSERVERS FOR NONLINEAR SYSTEMS

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ABSTRACT

Sliding controllers have recently been shown to feature excellent robustness and performance properties for specific classes of nonlinear tracking problems. This paper examines the potential use of sliding surfaces for observer design. A particular observer structure including switching terms is shown to have promising properties in the presence of modelling errors and sensor noise.

1. INTRODUCTION

The notion of a sliding surface (Filippov, 1960) has been investigated mostly in the Soviet literature (see Utkin, 1977 for a review), where it has been used to stabilize a class of non-linear systems. Although it theoretically features excellent robustness properties in the face of *parametric* uncertainty, classical sliding mode control presents several important drawbacks that severely limit its practical applicability. In particular, it involves large control authority and control *chattering*. Chattering is in general highly undesirable in practice (with a few exceptions, such as the control of electric motors using pulse width modulation), since it implies extremely high control activity, and further may excite high-frequency dynamics neglected in the course of modeling, such as resonant structural modes, neglected actuator time-delays, or sampling effects. These problems can be remedied by replacing the chattering control by a smooth control interpolation in a boundary layer neighboring a time-varying sliding surface (Slotine and Sastry, 1983) and monitoring the boundary layer width so as not to excite the high-frequency unmodeled dynamics (Slotine, 1984).

In this paper, we consider the dual problem of designing state *observers* using sliding surfaces. We show that, as can be expected, sliding observers potentially offer advantages similar to those of sliding controllers, in particular inherent robustness to parametric uncertainty and easy application to important classes of nonlinear systems. Further , contrary to the case of controller design, chattering issues in sliding observer design are only linked to numerical implementation rather than 'hard' mechanical limitations.

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Basic concepts on implicit dynamics using sliding surfaces are introduced in Section 2. Section 3 applies the development to the design of sliding observers for nonlinear systems in 'companion form', i.e. of the form

$$x^{(n)} = f$$

where f is a nonlinear, uncertain function of the system state $\mathbf{x}=[x,\dot{x},\cdots,x^{(n-1)}]^T$. In Section 4, the methodology is extended to general observable nonlinear systems. Section 5 discusses observability requirements and their relationship to sliding observers. Concluding remarks are offered in Section 6.

2. BASIC CONCEPTS

2.1 Sliding Surfaces

Let us first briefly summarize the basic idea of a sliding mode, linked to the potential advantages of using discontinous (switching) control laws. Consider the dynamic system:

$$x^{(n)}(t) = f(\mathbf{x};t) + b(\mathbf{x};t)u(t) + d(t) \tag{1}$$

where u(t) is scalar control input, x is the scalar output of interest, and $\mathbf{x}=[x,\dot{x},\cdots,x^{(n-1)}]^T$ is the state. In equation (1) the function f(**x**;t) (in general nonlinear) is not exactly known, but the *extent of the imprecision* $|\Delta f|$ *on* $f(\mathbf{x};t)$ *is upper bounded by a known continuous function of* X *and* t; similarly control gain b(**x**;t) is not exactly known, but is of known sign, and is bounded by known, continuous functions of **x** and t. Both f(**x**;t) and b(**x**;t) are assumed to be continuous in **x** . The disturbance *d(t)* is unknown but bounded in absolute value by a known continuous function of time. The control problem is to get the state **x** to *track a specific state* $\mathbf{x}_d=[x_d,\dot{x}_d,\cdots,x_d^{(n-1)}]^T$ *in the presence of model imprecision* on f(**x**;t) and b(**x**;t), *and of disturbances d(t).* For this to be achievable from time $t=0$ using a finite control u, we must assume:

$$\tilde{\mathbf{x}}|_{t=0} = 0 \tag{2}$$

where $\tilde{\mathbf{x}} := \mathbf{x} - \mathbf{x}_d = [\tilde{x},\dot{\tilde{x}},\cdots,\tilde{x}^{(n-1)T}]$ is the tracking error vector; this assumption shall be further discussed later. We define a *time-varying sliding surface S(t)* in the state-space R^n as $s(\mathbf{x};t) = 0$ with

$$s(\mathbf{x};t) := \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x}\,, \quad \lambda > 0 \tag{3}$$

where λ is a positive constant. Given initial condition (2), *the problem of tracking $\mathbf{x} \equiv \mathbf{x}_d$ is equivalent to that of remaining on the surface S(t) for all $t > 0$* – indeed $s \equiv 0$ represents a linear differential equation whose unique solution is $\tilde{x} \equiv 0$, given initial conditions (2). Now a sufficient condition for such positive invariance of S(t) is to choose the control law u of (1) such that outside of S(t)

$$\frac{1}{2}\frac{d}{dt}s^2(\mathbf{x};t) \leq -\eta|s| \tag{4}$$

where η is a positive constant. Inequality (4) constrains trajectories to point towards the surface S(t) (Figure 1), and is referred to as the *sliding condition.*

The idea behind equations (3, 4) is to pick-up a well-behaved function of the tracking error, s, according to (3), and then select the feedback control law u in (1) such that s^2 remains a Liapunov function of the closed-loop system *despite the presence of model imprecision and of disturbances.* Further, satisfying (3) guarantees that if condition (2) is not exactly verified, i.e. if $\mathbf{x}\,|_{t=0}$ is actually off $\mathbf{x}_d|_{t=0}$, the surface S(t) will none the less be reached in a finite time inferior to $|\,s(\mathbf{x}(0);0)\,|\,/\eta$, while definition (3) then guarantees that $\tilde{\mathbf{x}} \to 0$ *as* $t \to \infty$. Control laws that satisfy (4), however, have to be *discontinuous* across the sliding surface, thus leading in practice to control chattering.

The obvious problem in similarly exploiting sliding behavior in the design of *observers,* rather than controllers, is precisely that the full state is not available for measurement, and thus that a sliding surface definition analog to (3) is not adequate. Some intuition can be developed for addressing this difficulty by considering simple second-order dynamics.

2.2 Shearing Effect and Sliding Patches

Let us consider the generation of sliding behavior in a second-order system through input switching according to the value of a *single* component of the state, rather than a linear combination of both components, as in (3). The system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\,k_2\,sgn(x_1)$$

where k_2 is a positive constant and *sgn* is the sign function, clearly exhibits no sliding behavior (Figure 2). Instead, let us consider the system

$$\dot{x}_1 = x_2 - k_1 sgn(x_1)$$
$$\dot{x}_2 = -\,k_2 sgn(x_1)$$

where k_1 and k_2 are positive constants. The corresponding phase-plane trajectories are illustrated in Figure 3, which can be constructed from Figure 2 by shifting the trajectories on the right half-plane upwards, by the quantity k_1, and similarly shifting the left half-plane trajectories by $-k_1$. This shearing effect generates sliding behavior in the region

$$|x_2| \leq k_1 \tag{5}$$

which we shall refer to as the *sliding patch.*

Let us detail the analysis. The condition

$$\frac{d}{dt}(x_1)^2 < 0$$

is satisfied if condition (5) holds, which defines the sliding patch. The dynamics on the sliding patch itself can be derived from Filippov's solution concept (Filippov, 1960), which formalizes engineering intuition: the dynamics on the patch can only be a convex combination (i.e., an *average*) of the dynamics on each side of the discontinuity surface

$$\dot{x}_1 = \gamma(x_2 + k_1) + (1-\gamma)(x_2 - k_1)$$
$$\dot{x}_2 = \gamma\, k_1 + (1-\gamma)(-\,k_1)$$

The value of γ , and therefore the resulting dynamics, are then formally determined by the invariance of the patch itself:

$$\dot{x}_1 = 0 \quad => \quad \dot{x}_2 = -\,(k_2/k_1)\,x_2$$

Thus, x_2 exponentially decreases to 0 after reaching the sliding patch, with a time-constant k_1/k_2. Further, one can easily show that all trajectories starting on the x_2 axis reach the patch in a time smaller than $|x_2(t{=}0)|/(k_1k_2)$. Actually, sliding can be guaranteed from time t=0 by making k_1 and k_2 time-varying, with

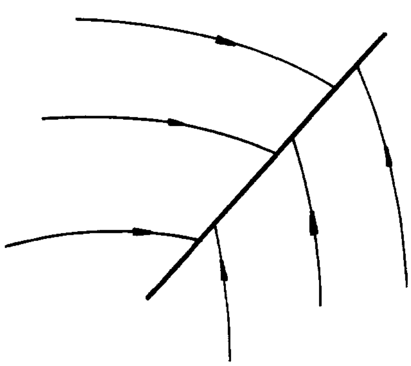

Figure 1: *The sliding condition*

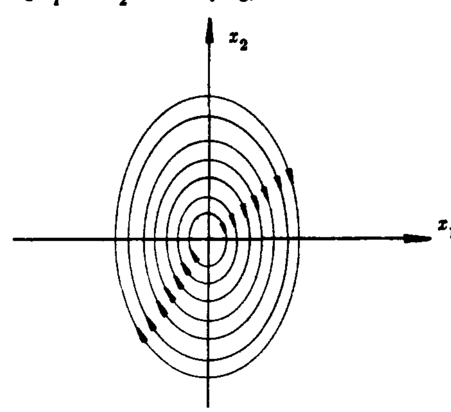

Figure 2: *Second order system with single input switching*

$$k_2/k_1 \geq a$$
$$k_1 > |x_2(t{=}0)|e^{-at}$$

where $a = a(t)$ is any positive function of time .

2.3 System Damping

Consider now the system

$$\dot{x}_1 = -\alpha_1 x_1 + x_2 - k_1 sgn(x_1)$$
$$\dot{x}_2 = -\alpha_2 x_1 - k_2 sgn(x_1)$$

Repeating the previous analysis, the sliding condition is verified in the extended region

$$x_2 \leq k_1 + \alpha_1 x_1 \qquad \text{if } x_1 > 0$$
$$x_2 \geq -k_1 + \alpha_1 x_1 \qquad \text{if } x_1 < 0$$

as illustrated in Figure 4. Thus, the addition of the damping term in α_1 increases the region of direct attraction. Further, the value of α_2 only affects the capture phase but not the dynamics on the patch itself, which remains unchanged:

$$\dot{x}_2 = -(k_2/k_1)\, x_2$$

3. IMPLICIT REDUCED-ORDER OBSERVERS FOR NONLINEAR SYSTEMS IN COMPANION FORM

3.1 Systems with a Single Measurement

Let us now consider the system

$$\ddot{x}_1 = f$$

where f is a nonlinear, uncertain function of the state $\mathbf{x} = [x_1, x_2 = \dot{x}_1]^T$, and let us exploit the preceding development to design an observer for this system, based on the measurement of x_1 alone. From the previous dicussion, we use an observer structure of the form

$$\dot{\hat{x}}_1 = -\alpha_1\tilde{x}_1 + \hat{x}_2 - k_1 sgn(\tilde{x}_1)$$
$$\dot{\hat{x}}_2 = -\alpha_2\tilde{x}_1 + \hat{f} - k_2 sgn(\tilde{x}_1) \qquad (6)$$

where $\tilde{x}_1 = \hat{x}_1 - x_1$, and the constants α_i are chosen as in a Luenberger observer (which would correspond to $k_1 = 0$, $k_2 = 0$) so as to place the poles of the linearized system at desired locations $-\varsigma_j$. The quantity $\hat{f}$ in (6) is the estimated value of f. The value of $\Delta f = \hat{f} - f$ depends both on the modelling effort and of the computational complexity allowable in the observer itself. In this paper, we assume that dynamic uncertainty Δf is explicitely bounded. Known nonlinear terms may also, for simplicity, be treated as bounded error (using known bounds on the actual system state) and included in Δf. The effect of Δf is compensated by exploiting this knowledge of its (generally time-varying) bound, as we shall later illustrate.

The resulting error dynamics can be written:

$$\dot{\tilde{x}}_1 = -\alpha_1\tilde{x}_1 + \tilde{x}_2 - k_1 sgn(\tilde{x}_1)$$
$$\dot{\tilde{x}}_2 = -\alpha_2\tilde{x}_1 + \Delta f - k_2 sgn(\tilde{x}_1)$$

The methodology can be directly extended to n^{th}-order systems in companion form:

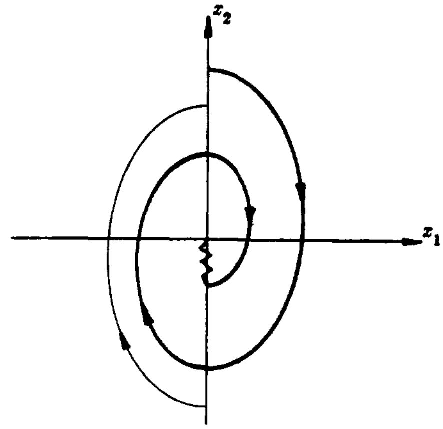

Figure 3: *Shearing effect*

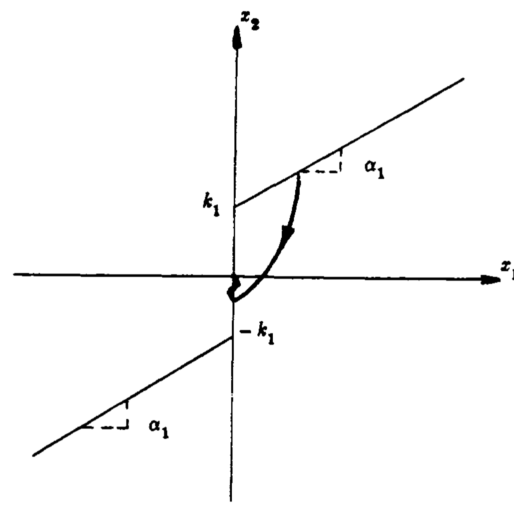

Figure 4: *Effect of damping on reachability*

$$x_1^{(n)} = f$$

where x_1 is the single measurement available. The observer structure is then of the form

$$\dot{\hat{x}}_1 = -\alpha_1\tilde{x}_1 + \hat{x}_2 - k_1 sgn(\tilde{x}_1)$$
$$\dot{\hat{x}}_2 = -\alpha_2\tilde{x}_1 + \hat{x}_3 - k_2 sgn(\tilde{x}_1)$$
$$\cdots$$
$$\dot{\hat{x}}_n = -\alpha_n\tilde{x}_1 + \hat{f} - k_n sgn(\tilde{x}_1)$$

The $n-1$ poles associated to the implicit dynamics on the patch are defined by

$$det(p\mathbf{I}_{n-1} - \begin{bmatrix} -(\alpha_2 + k_2/k_1) & 1 & 0 \ldots 0 \\ -(\alpha_3 + k_3/k_1) & 0 & 1 \ldots 0 \\ & & \ddots \\ -(\alpha_n + k_n/k_1) & 0 & 0 \ldots \begin{smallmatrix}1\\0\end{smallmatrix} \end{bmatrix}) = 0 \qquad (7)$$

Thus, the poles on the patch can be placed arbitrarily by proper selection of the ratios (k_i/k_1) , $[i = 2,...,n]$. A possible choice is to define k_1 as the desired precision in $\tilde{x}_2$, let

$$k_n \geq |\Delta f|$$

and in a constant ratio with k_1, and finally define the remaining poles k_i , $[i = 2,...,n-1]$ so that the implicit dynamics associated with the patch be critically damped, i.e., have all poles real and equal to a positive constant λ. One can then easily show that trajectories starting on the $\tilde{x}_1$ axis and in the sliding patch remain in the patch, and verify

$$|\tilde{x}_2^{(i)}| \leq (2\lambda)^i k_1 \qquad i = 0, \dots , n-2$$

from which the precision on $\hat{\mathbf{x}}$ can be derived.

A revealing remark can be made at this point on the relationship between the dynamics of the Luenberger part (defined by the ς_j , $[j = 1,...,n]$) and the implicit dynamics on the sliding patch (defined by the ratios (k_i/k_1) , $[i = 2,...,n]$) by considering the linear transformation

$$\tilde{\mathbf{z}} = \mathbf{M}\tilde{\mathbf{x}}$$

which makes damping explicit and puts the error dynamics in the form

$$\dot{\tilde{z}}_1 = -\varsigma_1\tilde{z}_1 + \tilde{z}_2 - k'_1 sgn(\tilde{z}_1)$$
$$\dot{\tilde{z}}_2 = -\varsigma_2\tilde{z}_2 + \tilde{z}_3 - k'_2 sgn(\tilde{z}_1)$$
$$\dots$$
$$\dot{\tilde{z}}_n = -\varsigma_n\tilde{z}_n + \Delta f - k'_n sgn(\tilde{z}_1)$$

with

$$\mathbf{k}' = \mathbf{M}\mathbf{k}$$

Indeed, assume that the desired Luenberger poles ς_j are such that

$$\varsigma_2 = \varsigma_3 = \dots = \varsigma_n =: \varsigma$$

and assume further that the implicit dynamics on the patch is required to be critically damped at the *same* break-frequency ς:

$$\lambda := \varsigma$$

Then one can easily verify that the required k'_i would simply be

$$k'_2 = 0 \ , \ k'_3 = 0 \ , \ \dots \ , \ k'_n = 0$$

In other words, in z-space, switching would only occur in the computation of $\dot{\tilde{z}}_1$ (although corresponding x-space switching would occur on all components), thus seeming to lead to rather uninteresting behavior. The importance of this observation can be fully understood only by a close examination of the observer's properties in the presence of measurement noise, which shall be discussed in the next section. Let us just remark for now that there is no reason for the implicit bandwidth on the patch to be identical to the bandwidth of the Luenberger part; in particular, reducing k_1 *increases* the bandwidth on the patch, yet may potentially *reduce* sensitivity to measurement noise by reducing the amplitude of the discontinuity in $\dot{\tilde{x}}_1$. It is this unusual nonlinear effect that we must try and exploit to make sliding observers superior to Luenberger observers or extended Kalman filters under certain noise conditions, as we now discuss.

3.2 Effects of Measurement Noise

Consider again a second order system with a single measurement, now corrupted by noise $v = v(t)$

$$\dot{\tilde{x}}_1 = -\alpha_1(\tilde{x}_1 + v) + \tilde{x}_2 - k_1 sgn(\tilde{x}_1 + v)$$
$$\dot{\tilde{x}}_2 = -\alpha_2(\tilde{x}_1 + v) + \Delta f - k_2 sgn(\tilde{x}_1 + v) \qquad (8)$$

Although the presence of the terms in $sgn(\tilde{x}_1 + v)$ makes an exact stochastic analysis fairly involved, useful insight can be obtained by using appropriate simplifying approximations.

Assume, first, that v is a deterministic C^1 signal of bounded spectrum:

$$0 \leq \omega < \omega_- \ \text{ or } \ \omega > \omega_+ \qquad => \qquad F_v(\omega) = 0$$

where F_v is the Fourier transform of v. Sliding behavior, if any, can then only occur on the surface

$$\tilde{x}_1 + v = 0$$

Repeating the analysis of Section 2.2, the sliding region is then defined by

$$|\tilde{x}_2 + \dot{v}| \leq k_1 \qquad (9)$$

and the equivalent dynamics are given by

$$\tilde{x}_1 = -v$$
$$\dot{\tilde{x}}_2 + (k_2/k_1)\, \tilde{x}_2 = -\, (k_2/k_1)\, \dot{v} + \Delta f$$

Two limiting cases deserve particular attention:

a) $\omega_+ \ << \ (k_2/k_1)$. We then have, if $\Delta f = 0$

$$\tilde{x}_2 \approx -\dot{v} \ << \ (k_2/k_1)\, v_{max}$$

In particular, the estimate of x_2 is exact if the measurement error in x_1 is constant.

b) $\omega_- \ >> \ (k_2/k_1)$. We then have, if $\Delta f = 0$

$$\tilde{x}_2 \approx 0$$

However, the bound on k_2/k_1 also implies that the observer's robustness to model uncertainty is directly limited by the value of ω_- . The corresponding precision in $\tilde{x}_2$ is then

$$|\tilde{x}_2| \ \leq \ r\, F \,/\, \omega_-$$

where $r \approx 3$ is the desired ratio between ω_- and (k_2/k_1), and F is the available (in general time-varying) bound on $|\Delta f|$. It is obtained by choosing k_1 and k_2 according to

$$k_1 \ \geq \ |\dot{v}| + rF/\omega_-$$
$$k_2 = k_1\omega_-/r$$

so as to satisfy (9) while maintaining k_2 larger or equal to F.

The above discussion implies that, as could be expected, the system cannot remain in a pure sliding mode in the presence of arbitrary measurement noise. Instead, assuming that the measurement noise is bounded by some constant v_0 , the system will remain in a vicinity of the x_2 axis of width v_0. The major potential advantage of the proposed sliding observer, over e.g. an extended Kalman filter, is then that the sliding observer can still be made considerably more robust to parametric uncertainty. This can be easily understood by considering the 'average' error of the observer, $\tilde{x}_a$ whose dynamics can be approximated as

$$\dot{\tilde{x}}_{1a} = -\alpha_1 \tilde{x}_{1a} + \tilde{x}_{2a} - k_1 \text{ Average}[\text{sgn}(\tilde{x}_{1a} + v)]$$
$$\dot{\tilde{x}}_{2a} = -\alpha_2 \tilde{x}_{1a} - k_2 \text{ Average}[\text{sgn}(\tilde{x}_{1a} + v)] + \Delta f$$

where Average$[\tilde{x}_{1a} + v]$ is computed over 'short' time periods during which $\tilde{x}_a$ is treated as a constant. If we assume for simplicity that $v(t)$ is white noise, then

$$\text{Average}[\text{sgn}(\tilde{x}_1 + v)] = \text{Expectation}[\text{sgn}(\tilde{x}_1 + v)]$$
$$= \int_{-\infty}^{+\infty} \text{sgn}(\tilde{x}_1 + v) p(v) dv = 2 \int_0^{\tilde{x}_1} p(v) dv$$

where the last equality assumes that probability density $p(v)$ is symmetric. Thus, the average value is an odd continuous function of $\tilde{x}_1$. For instance, if v is uniformly distributed on the interval $[-v_0, v_0]$, we get

$$\text{Expectation}[\text{sgn}(\tilde{x}_1 + v)] = \tilde{x}_1 / v_0$$

so that the average dynamics can be written

$$\dot{\tilde{x}}_{1a} = -(\alpha_1 + k_1/v_0)\tilde{x}_{1a} + \tilde{x}_{2a}$$
$$\dot{\tilde{x}}_{2a} = -(\alpha_2 + k_2/v_0)\tilde{x}_{1a} + \Delta f \tag{10}$$

Thus, *the effect of the switching terms is to modify the effective bandwidth of the average dynamics according to the actual level of the measurement noise.* In particular, we recover Fillipov's equivalent sliding dynamics as the noise level v_0 tends to zero.
Indeed,

$$1/v_0 \to \infty \quad => \quad \tilde{x}_{1a} \to 0 \quad => \quad \dot{\tilde{x}}_{1a} \to 0$$

so that

$$\dot{\tilde{x}}_{2a} \to -[(\alpha_2 + k_2/v_0)/(\alpha_1 + k_1/v_0)]\tilde{x}_{2a} + \Delta f$$
$$\to -(k_2/k_1)\tilde{x}_{2a} + \Delta f$$

The above simplified analysis can be used to guide the choice of the switching gains k_j. Consider, for instance, the average error dynamics of a third-order system

$$\dot{\tilde{x}}_{1a} = -(\alpha_1 + k_1/v_0)\tilde{x}_{1a} + \tilde{x}_{2a}$$
$$\dot{\tilde{x}}_{2a} = -(\alpha_2 + k_2/v_0)\tilde{x}_{1a} + \tilde{x}_{3a}$$
$$\dot{\tilde{x}}_{3a} = -(\alpha_3 + k_3/v_0)\tilde{x}_{1a} + \Delta f$$

still with uniform bounded white noise of amplitude v_0, and choose the α_i as in a standard Kalman filter. It is then reasonable to select the k_j so that the average dynamics be critically damped:

$$\alpha_1 + k_1/v_0 = 3\lambda$$
$$\alpha_2 + k_2/v_0 = 3\lambda^2$$
$$\alpha_3 + k_3/v_0 = \lambda^3$$

Further, the minimum acceptable value of λ is determined by the condition

$$|\tilde{x}_{1a}| \leq v_0$$

which can be written

$$v_0 \leq F/\lambda^3$$

where F is a constant (or 'slowly' varying, as compared to bandwidth λ) upper bound on Δf. The value of λ that yields the smallest k_j's is then

$$\lambda = (F/v_0)^{1/3} \tag{11}$$

which represents a reasonable choice as long as the corresponding k_j's remain positive. The bounds on $\tilde{x}_{2a}$ and $\tilde{x}_{3a}$ can be computed accordingly. In particular, they can be easily analysed in the frequency domain: letting p be the Laplace variable, we have

$$\tilde{x}_{2a} = [1 - (3\lambda^2 p + \lambda^3)/(p+\lambda)^3] (\Delta f/p^2)$$
$$\tilde{x}_{3a} = [1 - \lambda^3/(p+\lambda)^3] (\Delta f/p)$$

Thus, for the observer to be fully effective, λ must be larger than the frequency content of Δf, which, if tuning (11) is used, imposes in turn an upper bound on the noise level v_0. Alternatively, the condition can be satisfied by increasing λ to a value larger than (11), thus also increasing the k_j and therefore the noise content of the state estimates.

3.3 Implementation aspects

The reader may rightfully be concerned about the handling of the terms in $sgn(\tilde{x}_1 + v)$ in the numerical implementation of the proposed observer. The problem can be addressed by using a boundary layer approach similar to that of (Slotine, 1984), i.e., by replacing in the implementation $sgn(\tilde{x}_1 + v)$ by $sat[(\tilde{x}_1 + v)/\phi]$, where sat is the saturation function. One can then easily show that the results of the averaging analysis still hold provided that $\phi << v_0$. For such an interpolation to be effective, however, one should slightly 'color' the measurement noise by prefiltering the measurement of x_1, using e.g a first-order filter of bandwidth much larger than the frequency content of x_1. The sampling rate is then chosen accordingly to be consistent with the frequency of the prefiltering.

3.4 Systems with Multiple Measurements

The case of systems in companion form with multiple measurements is a particular instance of a more general class of systems, discussed in the next section.

4. EXTENSION TO GENERAL NONLINEAR OBSERVABLE SYSTEMS

Consider the n^{th} order nonlinear system :

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) \quad , \quad \mathbf{x} \in R^n$$

and, for convenience, consider a vector of measurements that are linearly related to the state vector :

$$\mathbf{z} = \mathbf{C}\mathbf{x} \quad , \quad \mathbf{z} \in R^p$$

We define an observer with the following structure :

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{f}}(\mathbf{x},t) - \mathbf{K}\mathbf{1}_s$$

where $\hat{\mathbf{x}} \in R^n$, $\hat{\mathbf{f}}$ is our *model* of $\mathbf{f}$, $\mathbf{K}$ is a $n \times p$ gain matrix to be specified, and $\mathbf{1}_s$ is the $p \times 1$ vector

$$\mathbf{1}_s = [sgn(\tilde{z}_1) \;\; sgn(\tilde{z}_2) \;\; sgn(\tilde{z}_3) \ldots sgn(\tilde{z}_p)]^T$$

where

$$\tilde{z}_i := \mathbf{c}_i\hat{\mathbf{x}} - z_i \quad (12)$$

and $\mathbf{c}_i$ is the corresponding row of the $p \times n$ $\mathbf{C}$ matrix. We also define the following error vectors :

$$\mathbf{s} := \tilde{\mathbf{z}} = \mathbf{C}(\hat{\mathbf{x}} - \mathbf{x}) \quad (13)$$

$$\tilde{\mathbf{x}} := \hat{\mathbf{x}} - \mathbf{x} \quad (14)$$

Using equations (8) and (9) we have :

$$\dot{\tilde{\mathbf{x}}} = \Delta\mathbf{f} - \mathbf{K}\mathbf{1}_s \quad (15)$$

where

$$\Delta\mathbf{f} := \hat{\mathbf{f}}(\hat{\mathbf{x}},t) - \mathbf{f}(\mathbf{x},t)$$

For convenience we can rewrite (15) as ,

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{f}} \;\; , \;\; \tilde{\mathbf{f}} = \Delta\mathbf{f} - \mathbf{K}\mathbf{1}_s \quad (16)$$

The p dimensional surface, $\mathbf{s} = \mathbf{0}$,will be *attractive* if ,

$$s_i\dot{s}_i < 0 \, , \quad i=1,...p \quad (17)$$

Sliding will occur on the surface if in arbitrarily small vicinity of $\mathbf{f}$,

$$s_i\dot{s}_i \leq -\eta|s_i| \, , \quad i=1,...p \quad (18)$$

Equations (18) define the *sliding region* , i.e. the multivariable extension of the *sliding patch* defined by (5). During sliding the system dynamics are effectively reduced from n^{th} order system to a $n-p$ equivalent or *reduced order* system .

The approximate dynamics on this reduced order manifold can be formally derived using the so-called 'equivalent control' method (Utkin,1977), which is equivalent to Fillipov's solution concept in the case of linear input switching . During sliding, the switching term in (16) is acting to keep $\mathbf{s} \equiv \mathbf{0}$, hence, formally, $\dot{\mathbf{s}} \equiv \mathbf{0}$. We can express the second condition as,

$$\text{grad}(\mathbf{s}).\bar{\mathbf{f}}(\tilde{\mathbf{x}},\bar{\mathbf{1}}_s) = \mathbf{0} \quad (19)$$

where

$$\bar{\mathbf{f}} := \Delta\mathbf{f} - \mathbf{K}\bar{\mathbf{1}}_s \quad (20)$$

and $\bar{\mathbf{1}}_s$ is the equivalent switching vector , which can be obtained from (13), (19) and (20) :

$$\mathbf{C}(\Delta\mathbf{f} - \mathbf{K}\bar{\mathbf{1}}_s) = \mathbf{0}$$

so that

$$\bar{\mathbf{1}}_s = (\mathbf{C}\mathbf{K})^{-1}\mathbf{C}\Delta\mathbf{f} \quad (21)$$

Thus , the equivalent dynamics on the reduced order manifold is given by :

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{I} - \mathbf{K}(\mathbf{C}\mathbf{K})^{-1}\mathbf{C})\Delta\mathbf{f}$$
$$\mathbf{C}\tilde{\mathbf{x}} = \mathbf{0} \quad (22)$$

<u>Example</u> <u>4.1</u> Companion form (single measurement)

Clearly the results of section 3.1 are a special case of equations (10) (without the addition of the linear Luenberger term) with the dynamics on the patch described by (22), where :

$$\mathbf{C} = [1\ 0\ 0\ ...\ 0]$$
$$\mathbf{k} = [k_1\ ....\ k_n]^T$$
$$\Delta\mathbf{f} = [\tilde{x}_2,\tilde{x}_3,...,\tilde{x}_n,\hat{f}(\hat{\mathbf{x}}) - f(\mathbf{x})]^T$$

Expression (22) then yields the reduced order dynamics previously obtained.

<u>Example</u> <u>4.2</u> Non-companion form (single measurement)

Consider the second order system,

$$\dot{x}_1 = f_1(x_1,x_2)$$
$$\dot{x}_2 = f_2(x_1,x_2)$$
$$z = x_1$$

Equation (10) can be written

$$\dot{\hat{x}}_1 = \hat{f}_1(\hat{x}_1,\hat{x}_2) - k_1 sgn(\tilde{x}_1)$$
$$\dot{\hat{x}}_2 = \hat{f}_2(\hat{x}_1,\hat{x}_2) - k_2 sgn(\tilde{x}_1)$$

In order to use equivalent dynamics (22) we identify :

$$\mathbf{C} = [1\ 0] \;\; , \;\; \mathbf{K} = [k_1\ k_2]^T \;\; , \;\; \Delta\mathbf{f} = [\Delta f_1\ \Delta f_2]^T$$

where $\Delta f_i = \hat{f}_i - f_i$. The sliding condition (17) becomes ,

$$\tilde{x}_1(\Delta f_1 - k_1 sgn(\tilde{x}_1)) < 0$$

Thus , sliding occurs when $\tilde{x}_1 = 0$ and $|\Delta f_1| < k_1$. Equation (22) yields the sliding dynamics

$$\tilde{x}_1 = 0$$
$$\dot{\tilde{x}}_2 = -(k_2/k_1)\Delta f_1 + \Delta f_2 \quad (23)$$

The structure of Δf_1 and Δf_2 must be known before any further analysis can be done, as we now discuss .

5. OBSERVABILITY REQUIREMENTS

Consider the system defined by :

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) \;\; , \;\; \mathbf{x} \in R^n$$
$$\mathbf{z} = \mathbf{g}(\mathbf{x},t) \;\; , \;\; \mathbf{z} \in R^p$$

This system must be observable in order for any observer structure to be succesfull in reconstructing the state $\mathbf{x}$ from the measurement $\mathbf{z}$. Convenient algebraic observability conditions on $\mathbf{f}$ and $\mathbf{g}$ are not nearly so easy to find as in the linear case. (Hermann and Krener,1977) discuss the use of Lie derivatives to develop local conditions. Intuitively, in order for the system to be observable one must be able to perform successive differential operations on $\mathbf{g}(\mathbf{x})$ until an implicit inversion can be performed to obtain $\mathbf{x}$.

Consider for instance the second order nonlinear system of Example 4.2 :

$$\dot{x}_1 = f_1(x_1,x_2)$$
$$\dot{x}_2 = f_2(x_1,x_2)$$
$$z = x_1$$

In order for this system to be observable, f_1 must be a single valued function of x_2. One can see from equation (23) that Δf_1 must be a function of $\tilde{x}_2$ in order for the control term $-(k_2/k_1)\Delta f_1$ to have any influence on the error dynamics .

In general, the observability condition is strongly linked to equation (22) through the structure of the $\Delta \mathbf{f}$ vector, and an unobservable system will result in uncontrollable error dynamics .

6. CONCLUDING REMARKS

But is it Art ? Clearly, this study is only a step in developing a complete and systematic methodology of sliding observer design for nonlinear systems, and the reader may not want to throw away his Kalman filters yet. However, sliding observers have intriguing properties, and in particular uncommon behavior in the presence of measurement noise, which should make them worthy of extensive further research.

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