Screw Theory

#3

The Screw Theory

1. Screw Theory?
   - The representation of the movement of the rigid body
     - ✓ For intuitiveness and calculation speed, we need a different method more than the conventional method based on cartesian space (x, y, z)
     - ✓ Unlike the conventional transfer matrix-based method, the screw theory represents the movement of the rigid body
   - The screw theory can be classified as :
     - ✓ Finite screw (Screw)
     - ✓ Instantaneous screw (Twist)

The Quaternion

2. Quaternion?

- The number made of scalar and vector which made of an imaginary numbers
- $\boldsymbol{h} = h_1 + h_2 i + h_3 j + h_4 k = (Re(h), Im(h))$
  - ✓ h : Quaternion, $Re(h)$: Real number, $Im(h)$: Imaginary numbers
  - ✓ Summation, subtraction, and conjugate act same as conventional real-imaginary number
- Multiplicity acts like this :

Quaternion multiplication table

1 \ 2	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

ij = -ji = k
jk = -kj = i
ki = -ik = j

$$\boldsymbol{h}^{mult} = \begin{bmatrix} h_0 & -h_1 & -h_2 & -h_3 \\ h_1 & h_0 & -h_3 & h_2 \\ h_2 & h_3 & h_0 & -h_1 \\ h_3 & -h_2 & h_1 & h_0 \end{bmatrix}$$

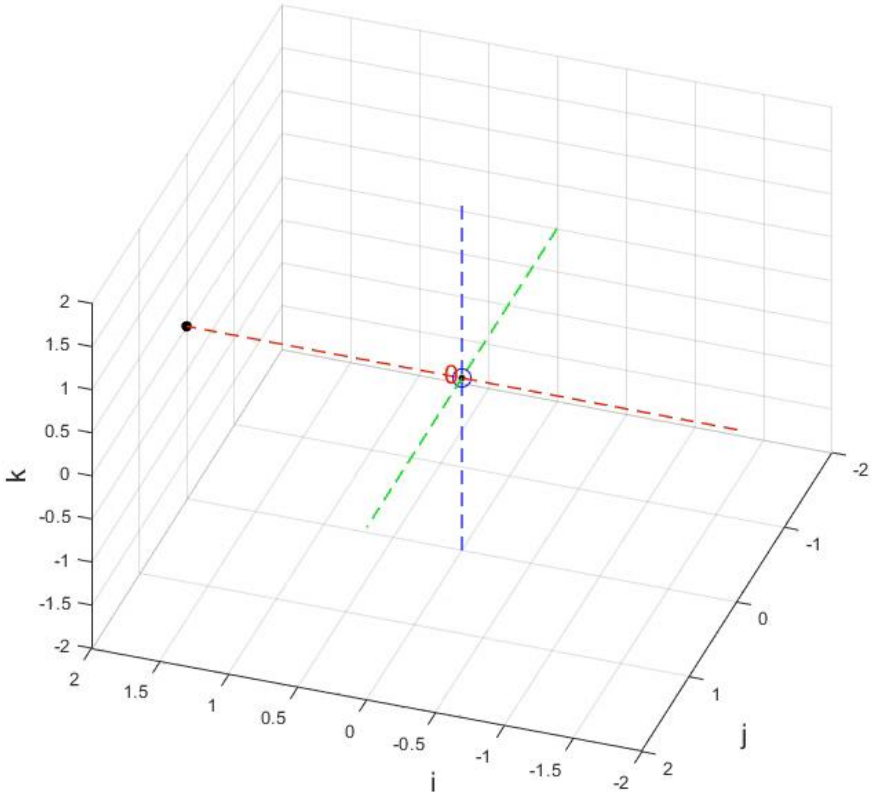
$$\boldsymbol{h} = [h_0 \quad h_1 \quad h_2 \quad h_3]^T$$

$$\boldsymbol{h}_1\boldsymbol{h}_2 = \boldsymbol{h}_1^{mult}\boldsymbol{h}_2$$

The Quaternion – can the transfer matrix be replaced?

- Quaternion
  - $h = h_1 + h_2 i + h_3 j + h_4 k = (Re(h), Im(h))$
    - ✓ h : Quaternion, $Re(h)$: Real number,
    - ✓ $Im(h)$: Imaginary numbers
  - How to rotate?
    - ✓ $\boldsymbol{h' = rhr^*}$
      - ➢ r^* is conjugate quaternion
    - ✓ $\boldsymbol{r = \cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\phi}{2}\right) n}$
      - ➢ ϕ is rotation angle
      - ➢ n is rotation axis vector
      - ➢ (form of imaginary vector)
    - ✓ $WHY\ \frac{\phi}{2}\ not\ just\ \phi$?
      - ➢ Check the video (The real part increases)

The number in middle of video is real part

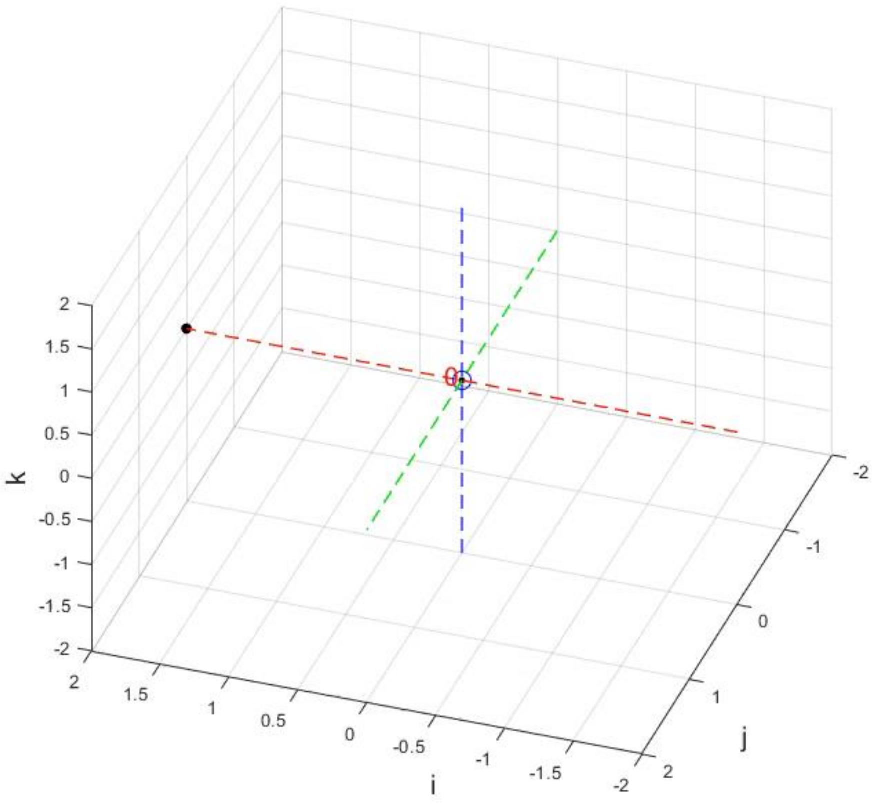


The Quaternion – can the transfer matrix be replaced?

- Quaternion
  - $h = h_1 + h_2 i + h_3 j + h_4 k = (Re(h), Im(h))$
    - ✓ h : Quaternion, $Re(h)$: Real number,
    - ✓ $Im(h)$: Imaginary numbers
  - This video moves h to $\acute{h}$
    - ✓ $h = 0 + 2i + 0j + 0k$
      $\acute{h} = 0 + 0i + 2j + 0k$
      by $n = [1, 1, 1]$
    - ✓ First trajectory shows
      - ➢ $\boldsymbol{\acute{h}_1 = rh}$, where $\boldsymbol{r}(\boldsymbol{\phi} = \left[\mathbf{0}, \frac{\mathbf{2\pi}}{\mathbf{3}}\right])$
    - ✓ Second trajectory shows
      - ➢ $\boldsymbol{\acute{h}_2 = \acute{h}_1 r^*}$, where $\boldsymbol{r}(\boldsymbol{\phi} = \left[\mathbf{0}, \frac{\mathbf{2\pi}}{\mathbf{3}}\right])$
    - ✓ Last trajectory shows
      - ➢ $\boldsymbol{\acute{h} = rhr^*}$

      where $\boldsymbol{r}\left(\boldsymbol{\phi} = \left[\mathbf{0}, \frac{\mathbf{2\pi}}{\mathbf{3}}\right]\right), \boldsymbol{r}^*\left(\boldsymbol{\phi} = \left[\mathbf{0}, \frac{\mathbf{2\pi}}{\mathbf{3}}\right]\right)$

The number in middle of video is real part



So we need to use quarternion?

Representing successive rotation using quaternion….

$$\acute{\boldsymbol{h}} = \cdots \boldsymbol{r_3 r_2 r_1 h r_1^* r_2^* r_3^*} \cdots$$

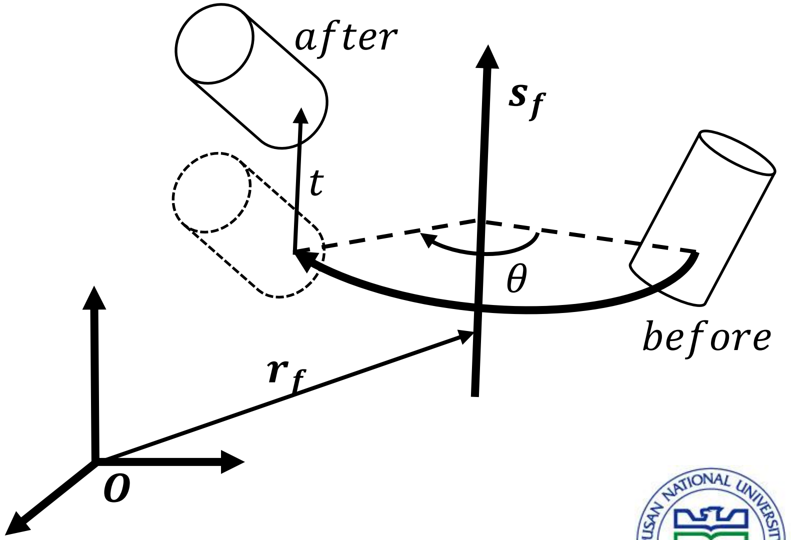
This makes the calculation complex, furthermore the position term is absent

By using screw theory, the calculation can be simple, including the motion of position.

The Quaternion to the finite screw

- Quaternion
  - $\boldsymbol{h} = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\boldsymbol{s_f}$
    - ✓ $\boldsymbol{h}$: Quaternion,
  - $\boldsymbol{L_f}$: Rotation axis (Lie group finite motion), but with Plücker Coordinates
    - ✓ $\boldsymbol{L_f} = (\boldsymbol{s_f^T}\ (\boldsymbol{r_f} \times \boldsymbol{s_f})^{\boldsymbol{T}})$ **<< which we want to use/express/calculate**
- Quaternion to the Dual Quaternion
  - $\boldsymbol{D = cos\left(\frac{\breve{\theta}}{2}\right) + sin\left(\frac{\breve{\theta}}{2}\right)\breve{S}}$
    - ✓ $\breve{\theta} = (\theta + \varepsilon t)$
    - ✓ $\breve{\boldsymbol{S}} = (\boldsymbol{s_f} + \varepsilon(\boldsymbol{r_f} \times \boldsymbol{s_f}))$
    - ✓ ε : dual number $(\varepsilon^2 = 0)$

$(\theta, t) \in \mathbb{R}, (\boldsymbol{s_f}, \boldsymbol{r_f}) \in \mathbb{R}^{3\times 1}$

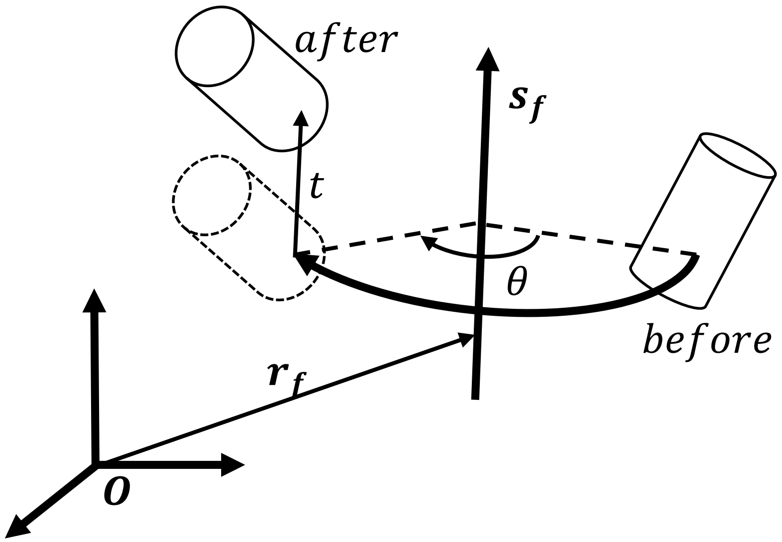


The Quaternion to the finite screw

- Quaternion to the Dual Quaternion
  - $\boldsymbol{D} = \boldsymbol{cos}\left(\frac{\breve{\theta}}{\mathbf{2}}\right) + \boldsymbol{sin}\left(\frac{\breve{\theta}}{\mathbf{2}}\right)\breve{\boldsymbol{S}}$
    - ✓ $\breve{\theta} = (\theta + \varepsilon t)$
    - ✓ $\breve{\boldsymbol{S}} = (\boldsymbol{s_f} + \varepsilon(\boldsymbol{r_f} \times \boldsymbol{s_f}))$
    - ✓ ε : dual number ($\varepsilon^2 = 0$)
  - $\boldsymbol{D} = \cos\left(\frac{\theta+\varepsilon t}{2}\right) + \sin\left(\frac{\theta+\varepsilon t}{2}\right)(\boldsymbol{s_f} + \varepsilon(\boldsymbol{r_f} \times \boldsymbol{s_f}))$
    - ✓ $\cos\left(\frac{\theta+\varepsilon t}{\mathbf{2}}\right) = \cos\left(\frac{\theta}{\mathbf{2}}\right) - \varepsilon\frac{t}{2}\sin(\frac{\theta}{2})$
    - ✓ $\sin\left(\frac{\theta+\varepsilon t}{\mathbf{2}}\right) = \sin\left(\frac{\theta}{\mathbf{2}}\right) + \varepsilon\frac{t}{2}\cos(\frac{\theta}{2})$
  - $\boldsymbol{D} = \cos\left(\frac{\theta}{\mathbf{2}}\right) - \varepsilon\frac{t}{2}\sin\left(\frac{\theta}{2}\right) + \left(\sin\left(\frac{\theta}{\mathbf{2}}\right) + \varepsilon\frac{t}{2}\cos\left(\frac{\theta}{2}\right)\right)\left(\boldsymbol{s_f} + \varepsilon(\boldsymbol{r_f} \times \boldsymbol{s_f})\right)$

    $= \left(\cos\left(\frac{\theta}{\mathbf{2}}\right) - \varepsilon\frac{t}{2}\sin\left(\frac{\theta}{2}\right)\right) + (\sin\left(\frac{\theta}{\mathbf{2}}\right)\boldsymbol{s_f} + \varepsilon(\sin\left(\frac{\theta}{\mathbf{2}}\right)(\boldsymbol{r_f} \times \boldsymbol{s_f}) + \frac{t}{2}\cos\left(\frac{\theta}{2}\right)\boldsymbol{s_f}))$

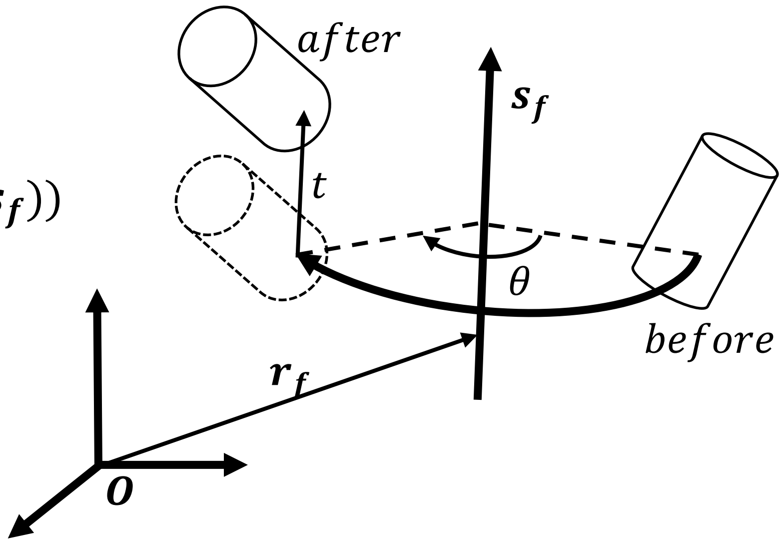
    $= D_s + \boldsymbol{D_v}$



$(\theta, t) \in \mathbb{R}, (\boldsymbol{s_f}, \boldsymbol{r_f}) \in \mathbb{R}^{3\times 1}$

The Quaternion to the finite screw

- Dual Quaternion to the Finite Screw
  - $\boldsymbol{D} = \cos\left(\frac{\theta}{2}\right) - \varepsilon\frac{t}{2}\sin\left(\frac{\theta}{2}\right) + \left(\sin\left(\frac{\theta}{2}\right) + \varepsilon\frac{t}{2}\cos\left(\frac{\theta}{2}\right)\right)\left(\boldsymbol{s_f} + \varepsilon(\boldsymbol{r_f} \times \boldsymbol{s_f})\right)$
    $= \left(\cos\left(\frac{\theta}{2}\right) - \varepsilon\frac{t}{2}\sin\left(\frac{\theta}{2}\right)\right) + \left(\sin\left(\frac{\theta}{2}\right)\boldsymbol{s_f} + \varepsilon\left(\sin\left(\frac{\theta}{2}\right)(\boldsymbol{r_f} \times \boldsymbol{s_f}) + \frac{t}{2}\cos\left(\frac{\theta}{2}\right)\boldsymbol{s_f}\right)\right)$
    $= D_s + \boldsymbol{D_v}$
  - $\widehat{\boldsymbol{D_v}} = \dfrac{\begin{bmatrix} \sin\left(\frac{\theta}{2}\right)\boldsymbol{s_f} \\ \sin\left(\frac{\theta}{2}\right)(\boldsymbol{r_f}\times\boldsymbol{s_f}) + \frac{t}{2}\cos\left(\frac{\theta}{2}\right)\boldsymbol{s_f}) \end{bmatrix}}{\frac{\cos\left(\frac{\theta}{2}\right)}{2}}$
    $= 2\tan\left(\frac{\theta}{2}\right)\begin{bmatrix} \boldsymbol{s_f} \\ (\boldsymbol{r_f} \times \boldsymbol{s_f}) + \frac{t}{2}\cos\left(\frac{\theta}{2}\right)\boldsymbol{s_f}) \end{bmatrix} + t\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{s_f} \end{bmatrix}$
    ....This becomes the Finite Screw
  - **Finite screw** $\boldsymbol{S_f} = 2\tan\left(\frac{\theta}{2}\right)\begin{bmatrix} \boldsymbol{s_f} \\ (\boldsymbol{r_f} \times \boldsymbol{s_f}) + \frac{t}{2}\cos\left(\frac{\theta}{2}\right)\boldsymbol{s_f}) \end{bmatrix} + t\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{s_f} \end{bmatrix}$



$(\theta, t) \in \mathbb{R}, (\boldsymbol{s_f}, \boldsymbol{r_f}) \in \mathbb{R}^{3\times1}$

... So, where is the D_s ?

D_s is still in somewhere, waiting for when calculate the successive finite screw

Finite screw and Instantaneous screw

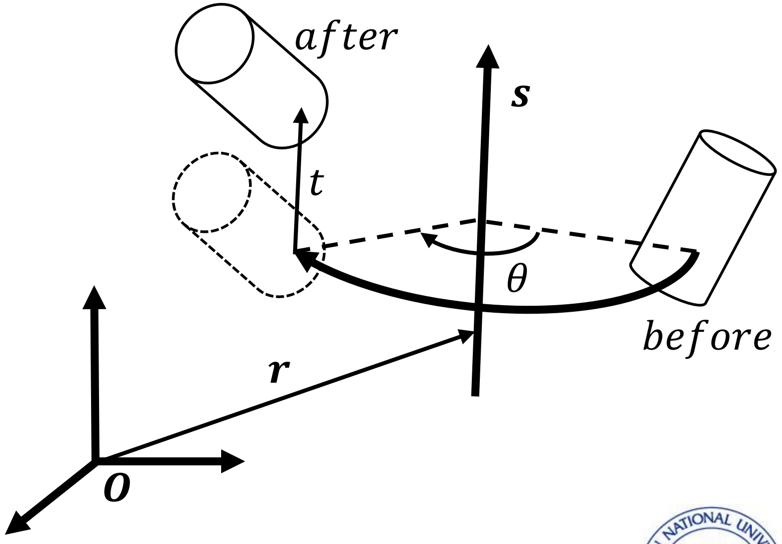
- Screw Theory represents the **shift of posture** from initial, contrast to DH parameter.
  - Can be categorized to the **Finite** and the **Instantaneous screw**.

$\boldsymbol{s} = \boldsymbol{s_f}$, $\boldsymbol{r} = \boldsymbol{r_f}$

1. **Finite screw** : $S_f = S_f(\theta, t, \boldsymbol{s}, \boldsymbol{r}) = 2\tan(\theta/2)[\boldsymbol{s}, \boldsymbol{r} \times \boldsymbol{s}]^T + t[\boldsymbol{0}_{1\times 3}, \boldsymbol{s}]^T$
   - Represents the **shift** of the posture.

2. **Instantaneous screw** : $S_i = S_i(\omega, v, \boldsymbol{s}, \boldsymbol{r}) = \omega[\boldsymbol{s}, \boldsymbol{r} \times \boldsymbol{s}]^T + v[\boldsymbol{0}_{1\times 3}, \boldsymbol{s}]^T$
   - **Assuming** $\boldsymbol{r}$ and $\boldsymbol{s}$ doesn't change.
   - Represents the **movement** of posture.
   - $\boldsymbol{\omega} = \dot{\theta}$
   - $\boldsymbol{v} = \dot{t}$

3. Relationship between the finite and instantaneous screw
   - $\dot{S_f} = \dot{S_f}(\dot{\theta}, \dot{t}, \dot{\boldsymbol{s}} = 0, \dot{\boldsymbol{r}} = 0)$
     $= \dot{\theta}/(\sec(\theta/2))^2\,[\boldsymbol{s}, \boldsymbol{r} \times \boldsymbol{s}]^T + \dot{t}[\boldsymbol{0}_{1\times 3}, \boldsymbol{s}]^T$
   - if $\boldsymbol{\theta} = \boldsymbol{0}$
   - $\dot{S_f} = S_i$



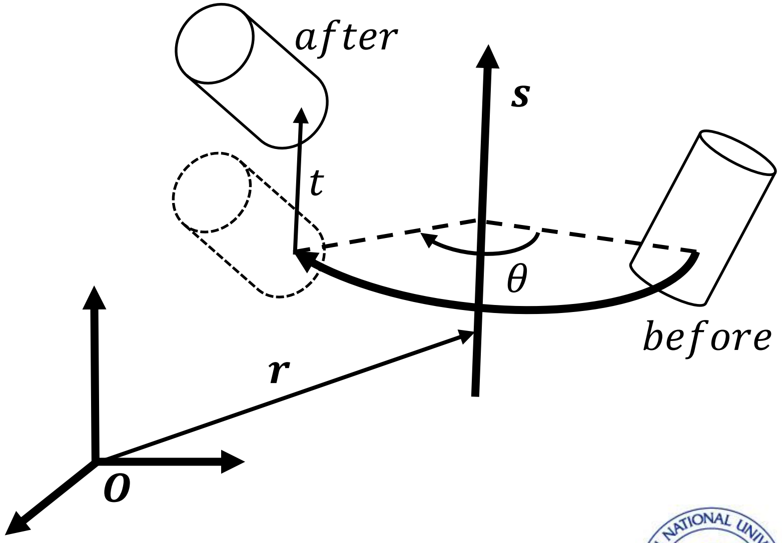
$(\theta, t) \in \mathbb{R}, (\boldsymbol{s}, \boldsymbol{r}) \in \mathbb{R}^{3\times 1}$

Relationship between finite screw and transfer matrix (tip)

Finite screw : $S_f = S_f(\theta, tr, \boldsymbol{s}, \boldsymbol{r}) = 2\tan{(\theta/2)}[\boldsymbol{s}, \boldsymbol{r} \times \boldsymbol{s}]^T + t[\boldsymbol{0}_{1\times3}, \boldsymbol{s}]^T$

Transfer matrix : $\begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix}$

- $\boldsymbol{R} = \boldsymbol{E_3} + \sin(\theta)\,\tilde{\boldsymbol{s}} + (1 - \cos(\theta))(\tilde{\boldsymbol{s}})^2$
  - ✓ **Rodrigues rotation formula**
- $\boldsymbol{t} = (\boldsymbol{E_3} - \boldsymbol{R})(r - \boldsymbol{s}^T\boldsymbol{r}\boldsymbol{s}) + t\boldsymbol{s}$



$(\theta, t) \in \mathbb{R}, (\boldsymbol{s}, \boldsymbol{r}) \in \mathbb{R}^{3\times1}$

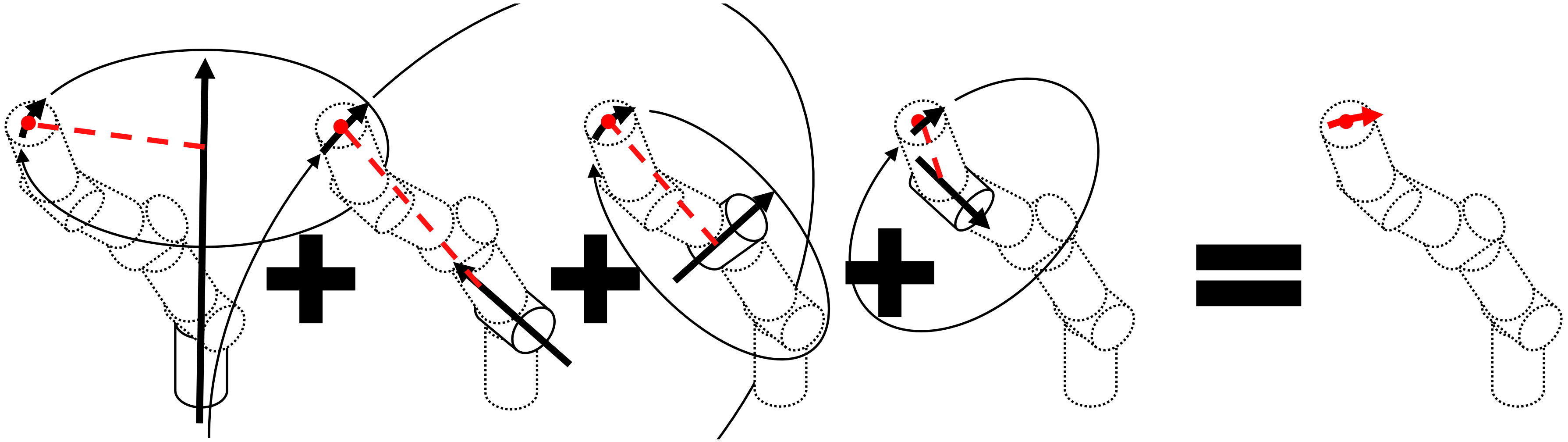
Characteristic features of Instantaneous Screw

Instantaneous Screw : $S_i = S_i(\omega, v, \boldsymbol{s}, \boldsymbol{r}) = \omega[\boldsymbol{s}, \boldsymbol{r} \times \boldsymbol{s}]^T + v[\boldsymbol{0}_{1\times3}, \boldsymbol{s}]^T$

- **Assuming** r and s doesn't change.
- Represents the **movement** of posture.

Mostly used for representing **movement of rigid body**, which **generated by successive active axes**

Because it has LINEAR PROPERTY, additivity can be applied.



$$S_{i,1}(\omega_1, 0, \boldsymbol{s_1}, \boldsymbol{r_1}) = \omega_1[s_1, \boldsymbol{r_1} \times \boldsymbol{s_1}]^T + S_{i,2}(\omega_2, 0, \boldsymbol{s_2}, \boldsymbol{r_2}) = \omega_2[s_2, \boldsymbol{r_2} \times \boldsymbol{s_2}]^T + S_{i,3}(\omega_3, 0, \boldsymbol{s_3}, \boldsymbol{r_3}) = \omega_3[s_3, \boldsymbol{r_3} \times \boldsymbol{s_3}]^T + S_{i,6}(\omega_6, 0, \boldsymbol{s_6}, \boldsymbol{r_6}) = \omega_6[s_6, \boldsymbol{r_6} \times \boldsymbol{s_6}]^T = S_{i,end}(\omega_{end}, 0, \boldsymbol{s_{end}}, \boldsymbol{r_{end}}) = \omega_{end}[s_{end}, \boldsymbol{r_{end}} \times \boldsymbol{s_{end}}]^T$$